

Differential operators and BV structures in noncommutative geometry

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Abstract. We introduce a new formalism of differential operators for a general associative algebra A . It replaces Grothendieck’s notion of differential operators on a commutative algebra in such a way that derivations of the commutative algebra are replaced by $\mathbb{D}\mathrm{er}(A)$, the bimodule of double derivations. Our differential operators act not on the algebra A itself but rather on $\mathcal{F}(A)$, a certain ‘Fock space’ associated to any noncommutative algebra A in a functorial way. The corresponding algebra $\mathcal{D}(\mathcal{F}(A))$ of differential operators is filtered and $\mathrm{gr}\,\mathcal{D}(\mathcal{F}(A))$, the associated graded algebra, is commutative in some ‘wheeled’ sense. The resulting ‘wheeled’ Poisson structure on $\mathrm{gr}\,\mathcal{D}(\mathcal{F}(A))$ is closely related to the double Poisson structure on $T_A\mathbb{D}\mathrm{er}(A)$ introduced by Van den Bergh. Specifically, we prove that $\mathrm{gr}\,\mathcal{D}(\mathcal{F}(A)) \cong \mathcal{F}(T_A(\mathbb{D}\mathrm{er}(A)))$, provided the algebra A is smooth.

Our construction is based on replacing vector spaces by the new symmetric monoidal category of *wheelspaces*. The Fock space $\mathcal{F}(A)$ is a commutative algebra in this category (a “commutative *wheelgebra*”) which is a structure closely related to the notion of wheeled PROP. Similarly, we have Lie, Poisson, etc., wheelgebras. In this language, $\mathcal{D}(\mathcal{F}(A))$ becomes the universal enveloping wheelgebra of a Lie wheelgebroid of double derivations.

In the second part of the paper we show, extending a classical construction of Koszul to the noncommutative setting, that any Ricci-flat, torsion-free bimodule connection on $\mathbb{D}\mathrm{er}(A)$ gives rise to a second order (wheeled) differential operator, a noncommutative analogue of the Batalin-Vilkovisky (BV) operator, that makes $\mathcal{F}(T_A(\mathbb{D}\mathrm{er}(A)))$ a BV wheelgebra.

In the final section, we explain how the wheeled differential operators $\mathcal{D}(\mathcal{F}(A))$ produce ordinary differential operators on the varieties of n -dimensional representations of A for all $n \geq 1$.

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1. Introduction

1.1. The general definition of differential operator on an abstract commutative algebra was introduced by Grothendieck. For noncommutative algebras, such as tensor algebras, Grothendieck’s definition still makes sense, but does not necessarily lead to a good notion (see Remark 2.1.3, 2.1.9 for details).

In this paper, we introduce a new notion of differential operators on associative algebras. Our approach is based on the observation that the tensor algebra of a vector space may be viewed as a *twisted* commutative algebra. The notion of twisted commutative algebra dates back at least to the 1950’s, appearing in algebraic topology, cf. also [Bar78]. One way to think about twisted commutative algebras is to interpret them as *commutative algebra objects* in the monoidal category of \mathbb{S} -modules, where an \mathbb{S} -module is a graded vector space equipped with symmetric group actions on its homogeneous components, cf. §2.3.

As a first step of our construction, we introduce a rather general definition of differential operators for a commutative algebra object in an *arbitrary* abelian symmetric monoidal category. Applying our definition in the special case of the monoidal category of \mathbb{S} -modules, one obtains quite a reasonable theory of differential operators on a twisted commutative algebra. Thus, for any twisted commutative algebra A , our construction produces a filtered algebra $\mathcal{D}(A)$ of differential operators. We show that $\mathrm{gr}\, \mathcal{D}(A)$, the associated graded algebra, is twisted commutative and, moreover, it has a natural structure of a twisted Poisson algebra (Theorem 2.5.1). Twisted Poisson structures on tensor algebras are related to Van den Bergh’s double Poisson structures [VdB08], as explained in [Sch09].

The theory of differential operators on twisted commutative algebras is not sufficient for the purposes of noncommutative geometry: it is essential to be able to consider differential operators on tensor algebras over a general *noncommutative* base. This amounts, formally, to replacing various tensor products over the ground field \mathbf{k} by tensor products over a noncommutative algebra A . If A is the path algebra of a quiver, for instance, tensoring over A may be interpreted as a gluing operation that joins heads and tails of various paths. Notice that joining the head and tail of the *same* path creates a “wheel,” i.e., an oriented cycle in the quiver. Thus, the next step of our construction is to introduce the concept of a *wheeled differential operator*.

The general point of view of this paper is that an adequate framework for doing noncommutative differential geometry is provided by the notion of wheelspace. Wheelspaces form a symmetric monoidal category, and there are natural notions of commutative, Poisson, and Lie algebra objects in that category, to be called commutative, Poisson, and Lie *wheelgebras*, respectively. Commutative wheelgebras turn out to be special cases of *wheeled PROPs*, introduced recently in [MMS09], which motivates the name. Given any noncommutative associative algebra A , we define a commutative wheelgebra, $\mathcal{F}(A)$, that resembles the Fock space construction. Having m inputs and n outputs for a wheeled PROP translates in our setting as having an $A^{\otimes m}$ - $A^{\otimes n}$ -bimodule structure.

We then apply our abstract categorical definition of differential operator to the special case of the monoidal category of wheelspaces. This way, any commutative wheelgebra \mathcal{W} gives rise to a filtered associative wheelgebra, $\mathcal{D}(\mathcal{W})$, of differential operators on \mathcal{W} . In particular, differential operators from $\mathcal{D}(\mathcal{F}(A))$ act on $\mathcal{F}(A)$ very much like Heisenberg algebras of creation and annihilation operators act on Fock spaces.

In the case that A is smooth, it turns out that the associated graded $\text{gr } \mathcal{D}(\mathcal{F}(A)) \cong \mathcal{F}(T_A \mathbb{D}\text{er}(A))$ (Theorem 3.6.7), and the induced Poisson wheelgebra structure is closely related to the double Poisson bracket introduced by Van den Bergh as a noncommutative counterpart of the canonical Poisson structure on T^*X , the total space of the cotangent bundle on a manifold X . This further explains the connection from [Sch09] between double Poisson structures and twisted Poisson structures on tensor algebras.

1.2. In the second half of the paper we transplant various differential geometric structures related to the notion of Batalin-Vilkovisky (BV) algebra and BV operator to noncommutative geometry. To explain this, first let X be a smooth (complex) manifold. It is well known that the Schouten-Nijenhuis bracket makes the space $\Lambda^\bullet T_X$ of polyvector fields on X a Gerstenhaber algebra.

Assume next that X is a Calabi-Yau manifold of dimension d . Any flat connection on the line bundle $\Lambda^d T_X$ gives rise to an odd second-order differential operator $D : \Lambda^\bullet T_X \rightarrow \Lambda^{\bullet-1} T_X$ such that $D^2 = 0$; see [Sch98]. The operator D , called the BV operator, satisfies the BV identity

$$(-1)^{|\xi|+1} \{\xi, \eta\} = D(\xi \wedge \eta) - D(\xi) \wedge \eta - (-1)^{|\xi|} \xi \wedge D(\eta), \quad (1.2.1)$$

for ξ, η local sections of $\Lambda^\bullet T_X$. Thus, D gives the Gerstenhaber algebra $\Lambda^\bullet T_X$ the structure of a BV algebra. Following Koszul [Kos85], we observe further that any Ricci-flat connection on the tangent bundle gives a flat connection on the line bundle $\Lambda^d T_X$, hence gives rise to a BV operator D as above.

In noncommutative geometry, vector fields are replaced by $\mathbb{D}\text{er}(A) := \text{Der}(A, A \otimes A)$, the bimodule of *double derivations* of a noncommutative algebra A ; see [CB99], [VdB08], [CBEG07]. The supercommutative algebra $\Lambda^\bullet T_X$ is therefore replaced by $T_A \mathbb{D}\text{er}(A)$, the tensor algebra of the bimodule $\mathbb{D}\text{er}(A)$. Note, however, that the component $\Lambda^d T_X$, of polyvector fields of top

degree, has no noncommutative counterpart. Thus, it is not clear *a priori* how to extend the Calabi-Yau geometry outlined above to the noncommutative setting.

With this in mind, we adapt Koszul’s approach and start with a *bi-module* connection ∇ on $\mathrm{Der}(A)$, as defined in [CQ95]. We study torsion and curvature for such connections, and also define the ‘trace of curvature’. Thus, we get the notion of a Ricci-flat bimodule connection on $\mathrm{Der}(A)$.

At this point we invoke our theory of differential operators on commutative wheelgebras. Our main construction associates with any Ricci-flat, torsion-free bimodule connection ∇ on $\mathrm{Der}(A)$ a second-order differential operator, D_∇ , on the commutative wheelgebra $\mathcal{F}(T_A \mathrm{Der}(A))$. Furthermore, we show that such an operator gives a BV wheelgebra structure that extends the double Schouten-Nijenhuis structure considered by Van den Bergh [VdB08].

1.3. The representation functor. It should be pointed out that the classical theory of differential operators on a commutative algebra is not a special case of our theory: differential operators on $\mathcal{F}(A)$ for a commutative algebra A are in general different from usual differential operators on A (although, if we view A itself as a commutative wheelgebra concentrated in degree zero, i.e., having no inputs or outputs, then differential operators on it in our sense are the same as in Grothendieck’s sense).

The relationship between commutative and noncommutative theories is provided by the notion of *representation functor*. Specifically, associated with a noncommutative algebra A , there is a sequence $\mathrm{Rep}_{\mathbf{d}}(A)$, $\mathbf{d} = 1, 2, \dots$, of affine schemes (in the conventional sense) parametrizing \mathbf{d} -dimensional representations of A . According to a philosophy advocated by Kontsevich, any reasonable associative (noncommutative) notion for A should go, via the representation functor, to the corresponding usual commutative notion. Furthermore, the two stories ‘merge’ asymptotically as $\mathbf{d} \rightarrow \infty$.

In §6 we extend the representation functor to act on wheelgebras and wheeled differential operators. This way, Theorem 5.7.1 ensures in particular that the action of wheeled differential operators corresponds, asymptotically, to the natural action of the quantized necklace Lie algebra from [Sch05, GS06]. Moreover, a noncommutative BV structure D_∇ on a smooth algebra A gives rise to an ordinary BV structure on each of the schemes $\mathrm{Rep}_{\mathbf{d}}(A)$, $\mathbf{d} \geq 1$.

We summarize our main results:

- Theorems 2.5.1 and 3.6.7, which extend the theory of commutative differential operators to the twisted-commutative and wheeled context, which apply to arbitrary associative algebras.
- Theorem 5.6.2, which proves the formula $D_\nabla^2 = i_{\mathrm{tr}(\nabla^2)}$ for smooth associative algebras, thus allowing us to deduce the equivalence of Ricci-flat torsion-free bimodule connections with wheeled BV structures.
- Theorem 5.7.1, which shows that path algebras of quivers are wheeled Calabi-Yau.

- Theorem 6.1.8, which explains that wheeled differential operators $\mathcal{D}(\mathcal{F}(A))$ map to actual differential operators under the representation functor.

1.4. Relation to necklace Lie bialgebras. Path algebras associated to quivers are free. Hence, they may be viewed as noncommutative analogues of flat affine spaces. The affine space comes equipped with the trivial connection on the tangent bundle, which is flat and torsion-free. Correspondingly, for a path algebra A , one has a trivial bimodule connection on $\mathrm{Der}(A)$, which is flat and torsion-free. The latter gives, by our construction, a wheeled BV operator D .

There is an alternative interpretation of the BV operator D for path algebras. To explain this, we remark first that an important class of (ordinary supercommutative) BV algebras comes from Lie bialgebras, as in, e.g., [Gin06], §2.10. We briefly explain this as follows.

Let \mathfrak{g} be a finite-dimensional involutive Lie bialgebra with cobracket $\delta: \mathfrak{g} \rightarrow \Lambda^2 \mathfrak{g}$. Given a basis x_1, \dots, x_n of \mathfrak{g} , one has structure constants for the Lie bracket defined by the equations $[x_i, x_j] = \sum_k c_{ij}^k \cdot x_k$ and structure constants for the cobracket defined by the equations $\delta(x_k) = \sum_{i,j} f_{ij}^k x_i \wedge x_j$.

Associated with such a Lie bialgebra, there is a BV operator on $\Lambda \mathfrak{g}$ defined by the formula

$$\begin{aligned} D : a_1 \wedge \cdots \wedge a_n \mapsto & \sum_{1 \leq k \leq n} (-1)^{k-1} \cdot \delta(a_k) \wedge a_1 \wedge \cdots \wedge \widehat{a}_k \wedge \cdots \wedge a_n \\ & + \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} \cdot [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \widehat{a}_i \wedge \cdots \wedge \widehat{a}_j \wedge \cdots \wedge a_n. \end{aligned} \quad (1.4.1)$$

In coordinates, the above formula reads

$$D = \sum_{i,j,k} f_{ij}^k \cdot x_i \wedge x_j \frac{\partial}{\partial x_k} + \sum_{i,j,k} c_{ij}^k \cdot x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \quad (1.4.2)$$

It turns out that the wheeled BV operator for a path algebra A may be viewed as a noncommutative analogue of the operator (1.4.2). Furthermore, this is not merely an analogy. The wheelspace-degree-zero part of the wheeled BV operator on $\mathcal{F}(A)$ is, in effect, given by (1.4.2) in the case of $\mathfrak{g} = A_{\mathrm{cyc}}$, where $A_{\mathrm{cyc}} := A/[A, A]$ is a super version of the necklace Lie bialgebra studied by one of us in [Sch05]. In the case of quivers with one vertex, the resulting BV algebra was studied by Barannikov ([Bar07], [Bar09, §1], and subsequent papers) in the context of quantum master equations and A_∞ algebras. There, the relevant BV algebra was denoted as F .

In more detail, let $\mathrm{pr} : A \rightarrow A_{\mathrm{cyc}}, a \mapsto [a]$ be the tautological projection. Further, following the strategy of [Sch07], §5.2 in the ordinary (not super) case, one may lift the necklace bracket and cobracket on A_{cyc} to maps

$$\{-, -\} : A \otimes A_{\mathrm{cyc}} \rightarrow A, \text{ and } \delta : A \rightarrow A \otimes A_{\mathrm{cyc}},$$

respectively. Then, the action of D on an element $u = a_1 \otimes \cdots \otimes a_n \otimes ([b_1] \cdots [b_m]) \in A^{\otimes n} \otimes \mathrm{SuperSym}_{\mathbf{k}}^m A_{\mathrm{cyc}}$ is given by the formula (if the a_i, b_j

are homogeneous)

$$\begin{aligned}
D(u) = & \sum_i \pm a_1 \otimes \cdots \otimes a_{i-1} \otimes \delta(a_i) \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes ([b_1] \cdots [b_m]) \\
& + \sum_{i,j} \pm \sigma'_{i,j} \{a_i, a_j\} a_1 \otimes \cdots \otimes \hat{a}_i \otimes \cdots \otimes \hat{a}_j \otimes \cdots \otimes a_n \otimes ([b_1] \cdots [b_m]) \\
& + \sum_{i,j} \pm a_1 \otimes \cdots \otimes a_{i-1} \otimes \{a_i, [b_j]\} \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes ([b_1] \cdots \widehat{[b_j]} \cdots [b_m]) \\
& + \sum_i \pm a_1 \otimes \cdots \otimes a_n \otimes \delta([b_i]) \cdot ([b_1] \cdots [b_{i-1}] \cdot [b_{i+1}] \cdots [b_m]) \\
& + \sum_{i,j} \pm a_1 \otimes \cdots \otimes a_n \otimes \{[b_i], [b_j]\} \cdot ([b_1] \cdots \widehat{[b_i]} \cdots \widehat{[b_j]} \cdots [b_m]),
\end{aligned} \tag{1.4.3}$$

with signs depending on degree and indices. The twisted-degree-zero part of the wheeled BV identity (and the equation $D^2 = 0$) for D is equivalent to the fact that $(A_{\text{cyc}}, \{-, -\}, \delta)$ is an involutive Lie bialgebra. The twisted-degree-one part of the BV identity was first discovered in [Sch07], §5.3; it reads:

$$\delta(ab) = \delta(a)(b \otimes 1) \pm (a \otimes 1)\delta(b) + (1 \otimes \text{pr})\{a, b\}. \tag{1.4.4}$$

We remark that the deformations of δ (also satisfying the above mentioned identities) described in [Sch07, §5.3] correspond, in our language, to connections other than the trivial one.

1.5. Future directions and related work. In the future, we hope to use wheeled differential operators to construct a *wheeled Heisenberg representation*, as well as a wheeled version of the Weil representation. We hope that the latter will have also a topological analogue, consisting of a projective action of the mapping class group of a Riemann surface on an appropriate wheeled Fock space, along the lines of [Gin06, §6.2]. Constructing such an action is a key ingredient in the approach to Calabi-Yau algebras arising from fundamental groups of aspherical 3-manifolds, as indicated in [Gin06], Conjecture 6.2.1.

Also, the formalism of noncommutative BV structures seems to play a role in trying to generalize the notion of Calabi-Yau algebra along the lines outlined in [EG07b], Remark 1.3.4.

Finally, in [Bar07], Barannikov constructs from any modular operad a BV-style master equation ($[\text{Bar07}, (5.5)]$) whose solutions are equivalent to algebras over the Feynman transform of that operad. When one sets the modular operad to be the operad denoted by $\mathbb{S}[t]$ in *op. cit.* (§9), one obtains a slight modification of the BV algebra defined above for the case of a quiver with one vertex, obtained by adding a parameter t and keeping track of a genus grading.

One may also form a directed analogue of the construction of [Bar07], replacing modular operads with wheeled PROPs (which include commutative

wheelgebras), by replacing undirected graphs with directed graphs. Here, one can additionally keep track of a genus grading at vertices.

Given a wheeled PROP P with differential d , and a finite-dimensional vector space V , one would obtain the same master equation as in *op. cit.*,

$$dm + zDm + \frac{1}{2}\{m, m\} = 0, \quad (1.5.1)$$

where m is now an element of

$$\sum_{i,j,k} P(i, j, k) \otimes_{\mathbf{k}[S_i \times S_j]} z^k \cdot \text{Hom}(V^{\otimes i}, V^{\otimes j}). \quad (1.5.2)$$

Here, $P(i, j, k)$ denotes the part of P with i inputs, j outputs, and with genus k , and z is a formal parameter which keeps track of the genus grading (which is *not* the same as the parameter t in the operad $\mathbb{S}[t]$ above). The space $\text{Hom}(V^{\otimes i}, V^{\otimes j})$ is associated to the part of the wheeled PROP with i inputs and j outputs, so that we can compose such operations by plugging an output of this into the input of another such operation, or vice-versa. The bracket

$$\{, \} : P(i, j, k) \otimes P(i', j', k') \rightarrow P(i + i' - 1, j + j' - 1, k + k') \quad (1.5.3)$$

now takes a sum over all ways to contract an input of an element in degrees (i, j, k) with an output of an element in degrees (i', j', k') , to obtain an element in degree $(i + i' - 1, j + j' - 1, k + k')$.

This is related to our bracket construction of §1.4 as follows. Consider a quiver Q with one vertex, and let V be the linear space generated by the arrows, Q . Consider V^* to be the linear space generated by the reverse of the arrows, Q^* . Then, we can then think of $\text{Hom}(V^{\otimes i}, V^{\otimes j}) \cong (V^*)^{\otimes i} \otimes V^{\otimes j}$ as spanned by a pair of an i -tuple of reverse edges and a j -tuple of edges. The bracket of two elements F, G in (1.5.2) sums over all ways of contracting an arrow of F with its reverse in G . This is analogous to the bracket on $\text{SuperSym}(\mathbf{k}\overline{Q})_{\text{cyc}}$ considered above (which, as we pointed out, is recovered when $P = \mathbb{S}[t]$ and $V = \mathbf{k}\overline{Q}$, except with an extra parameter t).

This bracket extends linearly to a map $P \otimes P \rightarrow P$. Similarly, the operator $D : P(i, j, k) \rightarrow P(i - 1, j - 1, k + 1)$ of (1.5.1) above takes a sum over all ways to contract an input with an output, and this also extends linearly to all of P . This operator D is a PROP analogue of the wheeled BV operator D discussed in this paper. In the one-vertex quiver setting of the previous paragraph, the operator D acts on an element F of (1.5.2) by summing over ways of contracting an arrow and a reverse arrow (which becomes the BV bracket of this section in the undirected setting with $P = \mathbb{S}[t]$ as discussed above).

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1.7. Notation. In the interests of self-containment, we repeat some standard definitions in noncommutative geometry. For a reference, see, e.g., [Gin05].

We will work over a fixed ground field \mathbf{k} of characteristic zero. The term “space” will mean “ \mathbf{k} -vector space.” The terms “map” or “operator,” unless otherwise specified, refer to any \mathbf{k} -linear maps. Unadorned tensor products will be taken to be over \mathbf{k} throughout.

We will use cycle notation for permutations: (a_1, a_2, \dots, a_n) means $a_1 \mapsto a_2 \mapsto a_3 \mapsto \dots \mapsto a_n$.

Notation 1.7.1. For any permutation $\sigma \in \Sigma_n$, we define $\tau_\sigma : V_1 \otimes V_2 \otimes \dots \otimes V_n \rightarrow V_{\sigma^{-1}(1)} \otimes V_{\sigma^{-1}(2)} \otimes \dots \otimes V_{\sigma^{-1}(n)}$ as the permutation of components corresponding to σ .

For any algebra A , define $A_{\text{cyc}} := A/[A, A]$, where $[A, A]$ is merely a vector space. When A is equipped with a grading and viewed as a superalgebra, $[\cdot, \cdot]$ will be the supercommutator.

For a finitely-generated projective A -module M , denote the dual module $\text{Hom}_A(M, A)$ by M^\vee .

For any algebra A over \mathbf{k} , we write A^{op} for the opposite algebra, and put $A^e := A \otimes A^{\text{op}}$. Note that an A^e -module is the same as an A -bimodule whose left and right actions, when restricted to \mathbf{k} , become the same. We will use the term “ A -bimodule” to refer to an A^e -module. The notation M^\vee will primarily be applied to A^e -modules.

For any A -bimodules M_1, M_2, \dots, M_n , one has a well defined outer A -bimodule action on $M_1 \otimes M_2 \otimes \dots \otimes M_n$. If M is any A -bimodule, then the space $\text{Hom}_{A^e}(M, A^e) = \text{Hom}_{A^e}(M, A \otimes A)$ is equipped with the structure of A -bimodule coming from the *inner* action on $A \otimes A$: namely, $(a \cdot \phi \cdot b)(m) = (1 \otimes a) \cdot \phi(m) \cdot (b \otimes 1)$. When we say that M is a projective A -bimodule, we mean projective as an A^e -module. In this case, $M^\vee := \text{Hom}_{A^e}(M, A^e)$.

Definition 1.7.2. Let $\mathbb{D}\text{er}(A) := \text{Der}(A, (A \otimes A))$ for any associative algebra A over \mathbf{k} . Elements of $\mathbb{D}\text{er}(A)$ are called “double derivations,” hence the notation.

Similarly to $\text{Hom}_{A^e}(M, A \otimes A)$, the space $\mathbb{D}\text{er}(A)$ is naturally an A -bimodule, with $(a \cdot \phi \cdot b)(x) = (1 \otimes a) \cdot \phi(x) \cdot (b \otimes 1)$.

Following [CQ95], let $\Omega^1 A := \ker(\mu_A)$ be the kernel of the multiplication map $\mu_A : A \otimes A \rightarrow A$, considered as an A -bimodule using outer multiplication. By [CQ95], there is a natural map $d : A \rightarrow \Omega^1 A$, and for any A -bimodule M , there is a natural isomorphism

$$d^* : \text{Hom}_{A^e}(\Omega^1 A, M) \xrightarrow{\sim} \text{Der}(A, M), \quad d^* \phi = \phi \circ d. \quad (1.7.3)$$

In particular, setting $M = A \otimes A$, one obtains $\text{Hom}_{A^e}(\Omega^1 A, A \otimes A) \cong \mathbb{D}\text{er}(A)$.

So, in the case that $\Omega^1 A$ is finitely-generated and projective, so is $\mathbb{D}\text{er}(A)$, and $\Omega^1 A$ and $\mathbb{D}\text{er}(A)$ are dual to each other. The finite-generation follows if we assume that A is finitely generated, and the fact that $\Omega^1 A$ is projective means precisely that A is smooth.

2. Differential operators in noncommutative settings

2.1. Grothendieck's definition of differential operators. Let A be a commutative algebra. We recall:

Definition 2.1.1. [Gro67] The only differential operator of order ≤ -1 is the zero operator.

Inductively, for any $n \geq 0$, a linear map $D : A \rightarrow A$ is said to be a differential operator of order $\leq n$ if for all $a \in A$, the operator $[D, a] : b \mapsto D(ab) - aD(b)$ is a differential operator of order $\leq n - 1$.

Equivalently, the above definition says that operators of order 0 are $\text{Hom}_A(A, A)$, the A -module homomorphisms from A to A .

There is an alternative definition to the above which coincides in the case of commutative algebras, obtained by the same inductive procedure except defining the order zero operators differently:

Definition 2.1.2. Differential operators of order zero are left-multiplication by elements of A , i.e., the image of A in $\text{Hom}_k(A, A)$. Then, inductively, a linear map $D : A \rightarrow A$ is a differential operator of order $\leq n$ if for all $a \in A$, the operator $[D, a] : b \mapsto D(ab) - aD(b)$ is a differential operator of order $\leq n - 1$.

In this paper, we extend the above definitions by using commutative algebras in more general tensor categories, and so both will coincide in these cases, and we don't need to prefer one over the other.

Remark 2.1.3. Definition 2.1.1 does not yield a good answer in the case $A = T_k V$, a free algebra with $\dim V \geq 2$: we claim that *all nonzero differential operators have order zero*, and are *right-multiplication by elements of A* . We explain this as follows.

First of all, for any A , D has order zero if and only if $[D, a] = 0$ for all a , which implies that $D(a) = D(a \cdot 1) = a \cdot D(1)$, for all a . Thus, D is right-multiplication by $D(1)$.

Next, differential operators of order 1 are those such that

$$[D, a](x) = x \cdot f(a), \quad \forall a, x, \quad (2.1.4)$$

for some fixed linear map $f : A \rightarrow A$. Assuming that $D(1) = 0$ by subtracting right-multiplication by $f(1)$, we get $D(a) = D(a \cdot 1) = f(a)$. Hence, (2.1.4) becomes

$$D(ab) = a \cdot D(b) + b \cdot D(a), \quad \forall a, b \in A. \quad (2.1.5)$$

Then, checking associativity, we have

$$D(abc) = D((ab)c) = ab \cdot D(c) + c \cdot D(ab) = ab \cdot D(c) + ca \cdot D(b) + cb \cdot D(a) \quad (2.1.6)$$

$$= D(a(bc)) = a \cdot D(bc) + bc \cdot D(a) = ab \cdot D(c) + ac \cdot D(b) + bc \cdot D(a), \quad (2.1.7)$$

so we deduce that

$$[a, c] \cdot D(b) + [b, c] \cdot D(a) = 0, \quad \forall a, b, c \in A. \quad (2.1.8)$$

Setting $b = c$, we get $[a, b] \cdot D(b) = 0$, for any $a, b \in A$. Thus, if $([A, A] \cdot x = 0) \Rightarrow (x = 0)$ (equivalently, $(\langle\langle [A, A] \rangle\rangle \cdot x = 0) \Rightarrow (x = 0))$,¹ then we deduce that $D = 0$, i.e., all differential operators of order ≤ 1 are of order 0, and inductively, *all* differential operators are of order 0. This condition holds in particular if A is not commutative and has no zero-divisors, such as $A = T_{\mathbf{k}}V$ for $\dim V \geq 2$. It also holds if A is a (deformed) preprojective algebra [Sch09, Example 2.11].

Remark 2.1.9. On the other hand, Definition 2.1.2 yields a more interesting space of differential operators in the case $A = T_{\mathbf{k}}V$. We claim that, in this case, the space of differential operators of order n is isomorphic to $\bigoplus_{m \leq n} \text{Der}(A)^{\otimes m}$, with action

$$\begin{aligned} & (D_1 \otimes D_2 \otimes \cdots \otimes D_m)(v_1 v_2 \cdots v_p) \\ &= \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq p} v_1 v_2 \cdots v_{i_1-1} D_1(v_{i_1}) v_{i_1+1} \cdots v_{i_2-1} D_2(v_{i_2}) \cdots \\ & \quad \cdots v_{i_m-1} D_m(v_{i_m}) v_{i_m+1} \cdots v_p. \end{aligned} \quad (2.1.10)$$

In words, the above formula roughly is the sum over all ways of applying D_1, D_2, \dots, D_m to distinct $v_{i_1}, v_{i_2}, \dots, v_{i_m}$, in left-to-right order (i.e., $i_1 < i_2 < \cdots < i_m$). Note that it is necessary here that v_1, \dots, v_p be elements of V rather than arbitrary elements of A .

To prove the above claim, first note that the above operations all have order $\leq n$: this follows from the identity

$$[(D_1 \otimes \cdots \otimes D_m), a](f) = D_1(a)(D_2 \otimes \cdots \otimes D_m)(f). \quad (2.1.11)$$

Next, any differential operator of order $\leq n$ is determined by its restriction to products of $\leq n$ generators. In the case of TV , this means the restriction to $V^{\otimes \leq n}$. The above operators realize all linear maps $V^{\otimes \leq n} \rightarrow TV$ uniquely, and hence (2.1.10) produces an isomorphism from $\bigoplus_{m \leq n} \text{Der}(TV)^{\otimes m}$ to the space of differential operators of order $\leq n$.

As a result of the observations of the previous paragraph, in this case of Definition 2.1.2 one deduces a description of the differential operators of order $\leq n$ on an arbitrary associative algebra A : if A is presented as a quotient of TV , then these are the subquotient of differential operators on TV of operators preserving the kernel of $TV \twoheadrightarrow A$, modulo operators whose image is in this kernel.

Remark 2.1.12. Even in the case of commutative algebras, the distinction between Definitions 2.1.1 and 2.1.2 can become important when one generalizes to the case of differential operators from an A -module M to itself; then, in the first definition, the order-zero differential operators are $\text{Hom}_A(M, M)$, while in the second they are the image of A in $\text{Hom}_{\mathbf{k}}(M, M)$.

¹The same condition was found in [Sch09], which is the condition under which a twisted Poisson algebra structure on $T_{\mathbf{k}}A$ (in the sense of this paper) must yield a double Poisson bracket on $A \otimes A$ (in the sense of [VdB08]).

2.2. Differential operators in symmetric monoidal categories. Below, we introduce a notion of differential operator on a commutative algebra object in an arbitrary symmetric monoidal category. In the case of the monoidal category of vector spaces our definition of differential operator reduces to the standard (Grothendieck's) definition. In this paper, we will be mostly interested in other monoidal categories such as the category of \mathbb{S} -modules, cf. §2.3, or the category of wheelspaces, cf. §3.1.

Let $(\mathcal{C}, \otimes, \beta)$ be a \mathbf{k} -linear symmetric monoidal category with product \otimes and braiding β , i.e., for any $X, Y \in \mathcal{C}$, we have $\beta_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$, which is functorial in X and Y . We will have in mind the case $\mathcal{C} = \mathbf{Vect}$ throughout, in which case one recovers Grothendieck's definition, as we will recall after each definition. For simplicity, we will omit all associativity isomorphisms, and pretend that $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$.² Let $C \in \mathcal{C}$ be a commutative algebra object in \mathcal{C} with multiplication map $\mu : C \otimes C \rightarrow C$.

Definition 2.2.1. Let $X \in \mathcal{C}$ be an object. A morphism $\phi : X \otimes C \rightarrow C$ is called *an action of X on C by differential operators of order $\leq n$* if the following is true:

For $n = -1$, the morphism must be the zero morphism;

and inductively on $n \geq 0$, $[\phi, \mu] : (X \otimes C) \otimes C \rightarrow C$ must be an action of $X \otimes C$ on C by differential operators of order $\leq n - 1$, where $[\phi, \mu] := \phi \circ (\text{Id}_X \otimes \mu) - \mu \circ (\text{Id}_C \otimes \phi) \circ (\beta_{X,C} \otimes \text{Id}_C)$.

Graphically, the two terms on the RHS are given by (the diagram is NOT commutative: rather the difference of the two compositions should be a differential operator of order one lower):

$$\begin{array}{ccccc}
 X \otimes C \otimes C & \xrightarrow{\text{Id}_X \otimes \mu} & X \otimes C & \xrightarrow{\phi} & C \\
 \beta_{X,C} \otimes \text{Id}_C \downarrow & & & \nearrow \mu & \\
 C \otimes X \otimes C & \xrightarrow{\text{Id}_C \otimes \phi} & C \otimes C & &
 \end{array} \tag{2.2.2}$$

Remark 2.2.3. The above definition also makes sense if C is not necessarily a commutative algebra in \mathcal{C} , but only an associative algebra. However, in this case, it might be better to use the following alternative definition which mimics Definition 2.1.2 rather than Definition 2.1.1. Define an action of X on C by differential operators of order zero to be a composition $\phi = \mu \circ (\iota \times \text{Id}) : X \otimes C \rightarrow C \otimes C \rightarrow C$. For $n \geq 1$, we then define actions by differential operators of order $\leq n$ by the same inductive definition as above. Note that, when C is a commutative algebra, this definition is equivalent to the above.

In the case $\mathcal{C} = \mathbf{Vect}$, the category of \mathbf{k} -vector spaces, C must be a commutative \mathbf{k} -algebra. Then, a vector space X together with an action by differential operators on C means first of all that $\phi : X \otimes C \rightarrow C$ is linear, i.e., we have a linear map $X \mapsto \text{End}_{\mathbf{k}}(C)$, $x \mapsto \phi_x$, with $\phi_x(c) := \phi(x \otimes c)$. In terms of ϕ_x , the two compositions in (2.2.2) become $\phi_x(ab)$ and $a\phi_x(b)$,

²This will result in no loss of generality. Moreover, it is a well known result of MacLane that any (symmetric) monoidal category is equivalent to a strictly associative one.

so ϕ is a differential operator of order n if and only if $\phi_x(ab) - a\phi_x(b)$ is a differential operator of order $n - 1$, for all $x \in X$ and all $a, b \in C$.

For any C -modules M, N in \mathcal{C} , one can similarly define a differential operator action $\phi : X \otimes M \rightarrow N$ of order $\leq n$ to be such that $[\phi, \mu] : (X \otimes C) \otimes M \rightarrow N$ is a differential operator action of order $\leq n - 1$, without changing anything else.

For any object X and an algebra object C , let $\mathcal{D}_{\mathcal{C}, \leq n}(X, C) \subset \text{Hom}_{\mathcal{C}}(X \otimes C, C)$ denote the vector space of actions of X on C by differential operators of order $\leq n$.

Definition 2.2.4. Define the *space of differential operators of order $\leq n$* to be an object $\mathcal{D}_{\leq n}(C) \in \mathcal{C}$ (if it exists) which represents the functor $\mathcal{C} \rightarrow \text{Vect}$, $X \mapsto \mathcal{D}_{\mathcal{C}, \leq n}(X, C)$.

By definition, whenever the object $\mathcal{D}_{\leq n}(C)$ exists, it comes equipped with an action $\psi_{\leq n} : \mathcal{D}_{\leq n} \otimes C \rightarrow C$, on C , by differential operators of order $\leq n$, such that the natural map $\text{Hom}_{\mathcal{C}}(X, \mathcal{D}_{\leq n}) \rightarrow \mathcal{D}_{\mathcal{C}, \leq n}(X, C)$ is an isomorphism for all $X \in \mathcal{C}$.

In the case of $\mathcal{C} = \text{Vect}$, we see that $\mathcal{D}_{\leq n}(C)$ is $\mathcal{D}_{\leq n}(C)$ is the usual space of differential operators of order $\leq n$ on C .

Remark 2.2.5. Similarly, one may define $\mathcal{D}_{\leq n}(M, N)$, for any C -modules $M, N \in \mathcal{C}$.

Proposition 2.2.6. (i) *If \mathcal{C} is abelian and has arbitrary limits, then there exists $(\mathcal{D}_{\leq n}(C), \psi_{\leq n})$ and it is unique up to unique isomorphism.*

(ii) *There are canonical inclusions $(\mathcal{D}_{\leq n}(C), \psi_{\leq n}) \hookrightarrow (\mathcal{D}_{\leq (n+1)}(C), \psi_{\leq (n+1)})$.*

Sketch of a proof. To prove (i), take the direct sum of all actions $X \otimes C \rightarrow C$ by differential operators of order $\leq n$, and mod by the kernel.

To prove (ii), note that any action by differential operators of order $\leq n$ is also an action of order $\leq n + 1$. This gives a morphism $j : \mathcal{D}_{\leq n}(C) \rightarrow \mathcal{D}_{\leq (n+1)}(C)$.

Assume next that $X := \ker j$ is a nonzero object. Then, the embedding $X \hookrightarrow \mathcal{D}_{\leq n}(C)$ gives a nonzero action of X on C by differential operators of order $\leq n$ which is, at the same time, the zero action of X on C by differential operators of order $\leq n + 1$. But the condition for an action by differential operators to be trivial (i.e., the action map is zero) does not depend on what order it is considered. We conclude that the morphism j must be injective. \square

Definition 2.2.7. Let \mathcal{C} be abelian with arbitrary limits, and $C \in \mathcal{C}$ a commutative algebra object. The differential operators $(\mathcal{D}(C), \phi)$ on C is defined to be the direct limit of all $(\mathcal{D}_{\leq n}(C), \psi_{\leq n})$ (if it exists).

As an example, let $\mathcal{C} = \text{Rep}_G$, the category of representations of a finite group G , and C a commutative algebra with a G -action, then we find that $\mathcal{D}(C)$ is the usual algebra of differential operators on C , equipped with the canonical G -action. In other words, as an ordinary filtered associative algebra, $\mathcal{D}(C)$ is identical with the usual Grothendieck construction, but it

comes equipped, by definition, with its G -structure. Similarly, one may say the same thing if G is an algebraic group and Rep_G is now understood as the category of $\mathbf{k}[G]$ -comodules. Note that, unlike in the case of Vect , in these examples it is essential that X be allowed to be an arbitrary object of \mathcal{C} and not merely the unit object (since not all differential operators are G -invariant).

If \mathcal{C} is the category of super vector spaces, and C is a supercommutative algebra, then we get the usual definition of the superalgebra of differential operators on C . Note that, in this case, we cannot use the original Grothendieck definition (taking C to live in Vect): for example, if $C = \mathbf{k}[\theta]/(\theta^2)$ is the supercommutative algebra with θ odd, then differential operators according to the original definition include only the even operators (i.e., operators of the form $\lambda + \mu\theta\partial_\theta$ for $\lambda, \mu \in \mathbf{k}$), and do not include ∂_θ itself. Similarly, our definitions apply in the case where \mathcal{C} is the category of dg-vector spaces.

Finally, given a commutative algebra A , let $\mathcal{C} := A\text{-mod}$ be the category of left A -modules, with monoidal structure given by the tensor product over A . Let C be a commutative algebra object in $A\text{-mod}$, that is a commutative A -algebra. In this case, our definition of $\mathcal{D}(C)$ gives the algebra of *relative* differential operators with respect to the projection $\text{Spec } C \rightarrow \text{Spec } A$.

2.3. Twisted algebras. We begin with the definitions of twisted commutative algebras (which date back to at least the 1950's in algebraic topology; see, e.g., [Bar78, Joy86]). The purpose is to provide a reasonable extension of differential operators to *tensor algebras* (which are clearly not commutative, but are twisted-commutative).

For the reader familiar with operads (by which we will always mean *linear* operads, i.e., operads \mathcal{O} such that $\mathcal{O}(n)$ is a vector space for all $n \geq 0$), a twisted algebra over an operad \mathcal{O} is the same as an \mathcal{O} -algebra in the symmetric monoidal category of \mathbb{S} -modules (also known as symmetric sequences of vector spaces). Recall that an \mathbb{S} -module is a $\mathbb{Z}_{\geq 0}$ -graded vector space $\mathcal{V} = \bigoplus_{m \geq 0} \mathcal{V}(m)$, equipped with an action of S_m on $\mathcal{V}(m)$ for all m . The category of \mathbb{S} -modules is a symmetric monoidal category, with $\mathcal{V} \otimes_{\mathbb{S}} \mathcal{W} = \bigoplus_{p \geq 0} (\mathcal{V} \otimes_{\mathbb{S}} \mathcal{W})(p)$ where $(\mathcal{V} \otimes_{\mathbb{S}} \mathcal{W})(p) := \bigoplus_{m=0}^p \text{Ind}_{S_m \times S_{p-m}}^{S_p} (\mathcal{V}(m) \otimes \mathcal{W}(p-m))$. The braiding is given by $\mathcal{Q} \otimes_{\mathbb{S}} \mathcal{P} \xrightarrow{\sim} \mathcal{P} \otimes_{\mathbb{S}} \mathcal{Q}$, $q \otimes p \mapsto (12)^{p,q}(p \otimes q)$. Moreover, any ordinary vector space may be viewed as an \mathbb{S} -module concentrated in degree zero, and hence we can tensor \mathbb{S} -modules by vector spaces, so viewed. Thus, any operad \mathcal{O} defines both a category of twisted algebras (\mathcal{O} -algebras in \mathbb{S} -modules) and a category of usual algebras (\mathcal{O} -algebras in vector spaces).

We give an explicit definition in terms of permutations, which, although a bit complicated, is important for computations.

Notation 2.3.1. For any $i_1, \dots, i_\ell \geq 1$, let $i_{i_1, \dots, i_\ell} : S_\ell \hookrightarrow S_{i_1 + \dots + i_\ell}$ be the monomorphism which sends a permutation $\sigma \in S_\ell$ to the permutation of $S_{i_1 + \dots + i_\ell}$ that permutes the cells of the partition $(\{1, \dots, i_1\}, \{i_1 + 1, \dots, i_1 + i_2\}, \dots, \{i_1 + \dots + i_{\ell-1} + 1, \dots, i_1 + \dots + i_\ell\})$, preserving the order in each cell of the partition, by rearranging all the cells according to σ .

We will frequently use the shorthand

$$\sigma^{i_1, \dots, i_\ell} := i_{i_1, \dots, i_\ell}(\sigma). \quad (2.3.2)$$

In particular, $(12)^{m,n}$ is the permutation $(1, 2, \dots, m, m+1, \dots, m+n) \mapsto (m+1, \dots, m+n, 1, \dots, n)$.

Definition 2.3.3. A twisted associative algebra $\mathcal{A} := \bigoplus_{m \geq 0} \mathcal{A}(m)$ is:

1. a graded associative algebra over \mathbf{k} with multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$, together with
2. an action of S_m on $\mathcal{A}(m)$ for all $m \geq 1$, such that
3. $\mu : \mathcal{A}(m) \otimes \mathcal{A}(n) \rightarrow \mathcal{A}(m+n)$ is a map of $S_m \times S_n \subset S_{m+n}$ -modules.

A twisted **commutative** algebra is a twisted associative algebra \mathcal{A} such that

$$(12)^{m,n} \mu(a \otimes b) = \mu(b \otimes a), \quad \forall a \in \mathcal{A}(m), b \in \mathcal{A}(n). \quad (2.3.4)$$

In other words, for a twisted commutative algebra we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{A}(m) \otimes \mathcal{A}(n) & \xrightarrow{\mu} & \mathcal{A}(m+n) \\ & \searrow \mu^{\text{op}} & \downarrow \sim (12)^{m,n} \\ & & \mathcal{A}(m+n). \end{array}$$

Remark 2.3.5. In general, a twisted algebra \mathcal{A} over an operad \mathcal{O} may be explicitly defined as follows: $\mathcal{A} = \bigoplus_{m \geq 0} \mathcal{A}(m)$, with an S_m action on each $\mathcal{A}(m)$ (i.e., \mathcal{A} is an \mathbb{S} -module), equipped with a map

$$\bigoplus_{m \geq 0} \mathcal{O}(m) \otimes_{S_m} (\mathcal{A}^{\otimes m}) \rightarrow \mathcal{A}, \quad (2.3.6)$$

which descends from maps

$$\mathcal{O}(m) \otimes \mathcal{A}(i_1) \otimes \dots \otimes \mathcal{A}(i_m) \rightarrow \mathcal{A}(i_1 + \dots + i_m) \quad (2.3.7)$$

of $S_{i_1} \times \dots \times S_{i_m} \subset S_{i_1 + \dots + i_m}$ -modules (under the standard embedding). Also, the map (2.3.6) is required to satisfy an associativity condition.

Any ordinary commutative algebra is an example of twisted commutative algebra concentrated in degree zero.

Another example of a twisted commutative algebra is the graded space $\bigoplus_{m \geq 0} \mathbf{k}[S_m]$ where multiplication is given by tensor product (using the standard inclusion $(S_i \times S_j) \subset S_{i+j}$) and where the S_n -action is given by *conjugation*.

For any \mathbb{S} -module \mathcal{M} , let $\text{Sym}_{\mathbb{S}} \mathcal{M}$ denote the symmetric algebra in the category of \mathbb{S} -modules generated by \mathcal{M} . Explicitly, we have

$$\text{Sym}_{\mathbb{S}} \mathcal{M} = \bigoplus_{m \geq 0} (\mathcal{M}^{\otimes m})_{S_m}. \quad (2.3.8)$$

This is a free twisted commutative algebra.

Notation 2.3.9. Given a vector space V and $i \in \{0, 1\}$, let $V_{\mathbb{S}, i}$ denote the \mathbb{S} -module concentrated in degree i with $(V_{\mathbb{S}, i})_i = V$.

It is clear that $\mathrm{Sym}_{\mathbb{S}}V_{\mathbb{S},0} = \mathrm{Sym}_{\mathbf{k}}V$ is the ordinary symmetric algebra of the vector space V . Similarly, one has $\mathrm{Sym}_{\mathbb{S}}V_{\mathbb{S},1} = T_{\mathbf{k}}V$ is the ordinary tensor algebra of the vector space V . Here, $T_{\mathbf{k}}V$ is equipped with the standard grading and the S_n -action on $V^{\otimes n}$ is given by permutation of components.

2.4. Differential operators on twisted-commutative algebras. Let $\mathcal{C} = \mathbb{S}\text{-mod}$ be the monoidal category of \mathbb{S} -modules. Then we may rewrite our general definition of differential operators in terms of individual operators as follows.

A morphism $X \otimes C \rightarrow C$ is an action of X on C by differential operators of order $\leq n$ iff, for all homogeneous $x \in X$, the map $C \xrightarrow{vs.} \{x\} \otimes C \rightarrow C$ is a *differential operator* $C \rightarrow C$ of degree $|x|$ and order $\leq n$, where the latter is defined as follows. First, an *operator* $C \rightarrow C$ of degree $m \geq 0$ is a degree- m morphism of graded vector spaces such that the restriction to $C_k \rightarrow C_{m+k}$ is a morphism of S_k -modules where the action on C_{m+k} is via $S_k \cong (\{\mathrm{Id}_{S_m}\} \times S_k) \hookrightarrow S_{m+k}$, i.e., the composition has the property that $\sigma \mapsto \sigma', \sigma'(i) = \begin{cases} i, & \text{if } i \leq m, \\ m + \sigma(i - m), & \text{otherwise.} \end{cases}$

Next, a differential operator of degree m and order ≤ -1 is the zero morphism. Inductively, $D : C \rightarrow C$ is a differential operator of degree m and order $\leq n$ iff, for all homogeneous $c \in C$, $[D, c] : C \rightarrow C$ is a differential operator of degree $m + |c|$ and order $\leq n - 1$. Here, $[D, c]$ sends any homogeneous $a \in C$ to $D(ca) - (21)^{|c|,m,|a|}cD(a)$, and extends linearly.

The above theory works better than Grothendieck's original definition on tensor algebras. As opposed to Remark 2.1.3, we have the following classification of differential operators in the *twisted-commutative* sense on $T_{\mathbf{k}}V$:

Proposition 2.4.1. *The twisted associative algebra $\mathcal{D}(A)$ of differential operators on $A = T_{\mathbf{k}}V$ is the quotient of the free twisted associative algebra $T_{\mathbb{S}}(\mathrm{End}(V)_{\mathbb{S},0} \oplus V_{\mathbb{S},1})$ by the commutation relations (where $[a, b] := ab - \sigma^{|a|,|b|}ba$ is the **twisted** commutator): for all $v, w \in V$ and $\phi, \psi \in \mathrm{End}(V)$,*

$$[\lambda_v, \lambda_w] = 0, \quad [D_\phi, \lambda_v] = \lambda_{\phi(v)}, \quad [D_\phi, D_\psi] = D_{[\phi, \psi]}, \quad (2.4.2)$$

where, for each $\phi \in \mathrm{End}(V)$, we denote by D_ϕ the corresponding image in $T_{\mathbb{S}}(\mathrm{End}(V)_{\mathbb{S},0} \oplus V_{\mathbb{S},1})$, which acts as a derivation on V by applying ϕ , and we denote by λ_v , for each $v \in V$, the operator of left-multiplication by v , which is the summand of $V_{\mathbb{S},1}$ in $T_{\mathbb{S}}(\mathrm{End}(V)_{\mathbb{S},0} \oplus V_{\mathbb{S},1})$.

In other words, the proposition may be interpreted as saying that $\mathcal{D}(A)$ is the universal enveloping twisted associative algebra of the twisted Lie algebra $\mathrm{End}(V)_{\mathbb{S},0} \oplus V_{\mathbb{S},1}$ with bracket (2.4.2).

Proof. Inductively, a differential operator of order $\leq n$ on $T_{\mathbf{k}}V$ is determined by its restriction to $V^{\otimes \leq n}$. By compatibility with permutations, this restriction must be describable using polynomials of degree $\leq n$ in operators of the form D_ϕ , together with tensoring everything on the left by linear combinations of elements $a_1 \otimes \cdots \otimes a_\ell$. Such an element is easily verified to be a differential operator. From this, the above description easily follows. \square

2.5. Differential operators are almost-commutative. By *almost-commutative*, we will always mean filtered associative with a commutative associated graded.

Let \mathcal{C} be any abelian symmetric monoidal category with arbitrary limits. Parallel to the case of vector spaces, we have

Theorem 2.5.1. *The differential operators $\mathcal{D}(C)$ have the canonical structure of filtered algebra in the category \mathcal{C} . Moreover, the associated graded $\mathrm{gr} \mathcal{D}(C)$ is a commutative algebra in \mathcal{C} . The commutator on $\mathcal{D}(C)$ induces a Poisson algebra structure in \mathcal{C} on $\mathrm{gr} \mathcal{D}(C)$.*

Here, a *filtered algebra object* in \mathcal{C} is an object $A \in \mathcal{C}$ equipped with a filtration in \mathcal{C} , $0 \subset A_{\leq 0} \subset A_{\leq 1} \subset \cdots$, such that $A_{\leq i} \cdot A_{\leq j} \subset A_{\leq i+j}$ for all i, j . The associated graded of an almost-commutative algebra in \mathcal{C} is *always* Poisson (via the commutator), so the final assertion of the theorem is tautological.

Sketch of a proof. By definition, $\mathcal{D}(C)$ is a filtered object of \mathcal{C} . We have to exhibit a multiplication $\circ : \mathcal{D}(C) \otimes \mathcal{D}(C) \rightarrow \mathcal{D}(C)$ in \mathcal{C} , show that it respects the filtration, and that the commutator $D \circ D' - D' \circ D$ of operators of order $\leq n, n'$ has order $\leq n + n' - 1$.

These can be proved using the inductive definition of differential operators, similarly to the case where $\mathcal{C} = \mathbf{Vect}$. Let $D : X \otimes C \rightarrow C$ and $D' : X' \otimes C \rightarrow C$ be differential operators. Then, one has the following

Lemma 2.5.2. *Let $(\mathrm{ad} \mu)(f) := [f, \mu]$ for any operator (see Definition 2.2.1). Then, as operators $(X \otimes X' \otimes C^{\otimes p}) \otimes C \rightarrow C$, we have*

$$(\mathrm{ad} \mu)^p(D \circ D') = \sum_{m+n=p} (\mathrm{ad} \mu)^m(D) \circ (\mathrm{Id} \otimes (\mathrm{ad} \mu)^n(D')) \circ \beta^{(m)}, \quad (2.5.3)$$

where

$$\beta^{(m)}(X \otimes X' \otimes C^{\otimes p} \otimes C) = (X \otimes C^{\otimes m}) \otimes (X' \otimes C^{\otimes n}) \otimes C \quad (2.5.4)$$

is the composition of braidings which preserves the left-to-right order of the factors of C .

The proof of the lemma is the same as in the case $\mathcal{C} = \mathbf{Vect}$, where it becomes (again using the right adjoint action $(\mathrm{ad} a)(D) = [D, a]$):

$$[\cdots [D \circ D', a_1], a_2], \cdots, a_p] = \sum_{I \subset \{1, \dots, p\}} \left(\prod_{i \in I} \mathrm{ad} a_i \right) (D) \circ \left(\prod_{j \in \{1, \dots, p\} \setminus I} \mathrm{ad} a_j \right) (D'). \quad (2.5.5)$$

We omit the proof.

The lemma shows that the filtration is multiplicative (we get a filtered algebra). To show almost-commutativity, we only need to notice that two operators of order zero must commute with each other, since C is a commutative algebra in \mathcal{C} . \square

2.6. Twisted Poisson algebras. The notion of twisted Poisson algebra is closely related to Van den Bergh's double Poisson algebras, as will be explained in Section 3.5 (cf. [Sch09]). As before, a twisted Poisson algebra is the same as an algebra, in the category of \mathbb{S} -modules, over the Poisson operad. We give, however, a more explicit definition:

Definition 2.6.1. A twisted (commutative)³ Poisson algebra is a twisted commutative algebra $A = \bigoplus_{m \geq 0} A[m]$ with a graded bracket $\{, \} : A \otimes A \rightarrow A$ satisfying

1. $\{, \} : A[m] \otimes A[n] \rightarrow A[m+n]$ is a map of $S_m \times S_n \subset S_{m+n}$ -modules,
2. $\{, \}$ is a twisted Lie bracket: For $a \in A[m], b \in A[n], c \in A[p]$,

$$\{a, b\} = -(12)^{n,m} \{b, a\}, \quad (2.6.2)$$

$$\{\{a, b\}, c\} + (132)^{p,m,n} \{\{c, a\}, b\} + (123)^{n,p,m} \{\{b, c\}, a\} = 0; \quad (2.6.3)$$

3. $\{, \}$ is twisted Poisson: this means

$$\{a, bc\} = \{a, b\}c + (12)^{n,m,p} b\{a, c\}. \quad (2.6.4)$$

As a consequence of Proposition 2.5.1, we have

Corollary 2.6.5. *Let A be a filtered twisted associative algebra with twisted commutative $\text{gr } A$. Then $\text{gr}(A)$ has a natural twisted Poisson algebra structure.*

Proposition 2.6.6. *For $A = T_{\mathbf{k}}V$, $\text{gr } \mathcal{D}(A) \cong \text{Sym}_{\mathbb{S}-\text{mod}}(\text{End}(V)_{\mathbb{S},0} \oplus V_{\mathbb{S},1})$, with the twisted Poisson bracket given by*

$$\{\lambda_v, \lambda_w\} = \{D_\phi, D_\psi\} = 0, \quad \{D_\phi, \lambda_v\} = \phi(v). \quad (2.6.7)$$

The proof is straightforward.

3. Differential operators on associative algebras and wheelgebras

The main goal of this section is to extend the formalism of differential operators from $T_{\mathbf{k}}M$ (for which the previous section is applicable) to $T_A M$, where A is an associative algebra and M is an A -bimodule. This requires introducing a new algebraic structure called a *wheelgebra*, which will be fundamental to the differential geometry of associative algebras. Precisely, we will define, for any associative algebra A , a *commutative wheelgebra* $\mathcal{F}(A)$, and define differential operators on A in terms of categorical differential operators on $\mathcal{F}(A)$.

³For us, Poisson algebras are commutative by definition.

3.1. Wheelspaces. Given an associative algebra A , one can use multiplication in A to construct, for each $m \geq 1$, various *contraction* maps $A^{\otimes m} \rightarrow A^{\otimes(m-1)}$. Abstract properties of such contraction maps are conveniently formalized in the following

Definition 3.1.1. A *wheelspace* is a $\mathbb{Z}_{\geq 0}$ -graded vector space $\mathcal{W} = \bigoplus \mathcal{W}(m)$, with $S_m \times S_m$ -actions on $\mathcal{W}(m)$, and contraction maps

$$\mu_{i,j} : \mathcal{W}(m) \rightarrow \mathcal{W}(m-1) \quad (3.1.2)$$

which are morphisms of $S_{m-1} \times S_{m-1}$ -modules, where the action on the left is by permuting the sets $(\{1, \dots, m\} \setminus \{j\}) \times (\{1, \dots, m\} \setminus \{i\})$, satisfying the following condition:

$$\mu_{i,j} = \mu_{i,1} \circ ((1, 2, \dots, j) \times (1, 2, \dots, i)). \quad (3.1.3)$$

Also, the contraction maps must satisfy the associativity condition

$$\mu_{i,j} \circ \mu_{k,\ell} = \mu_{k',\ell'} \circ \mu_{i',j'}, \quad (3.1.4)$$

where, if $i < k$, then $(i', k') = (i, k-1)$, and otherwise $(i', k') = (i+1, k)$; the same relationship holds between (j, ℓ) and (j', ℓ') .

Informally, we think of elements $w \in \mathcal{W}(m)$ as represented by a picture as in Figure 1, with a total of m inputs and m outputs. The contraction op-

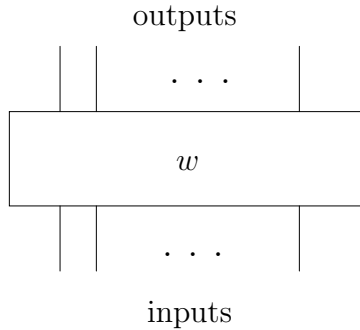
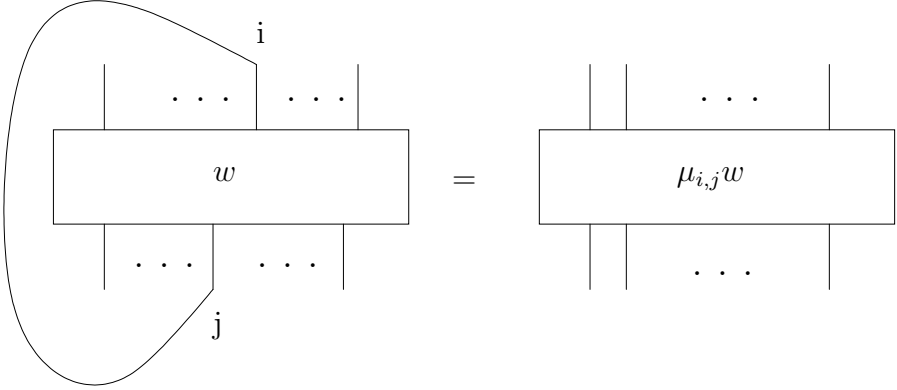
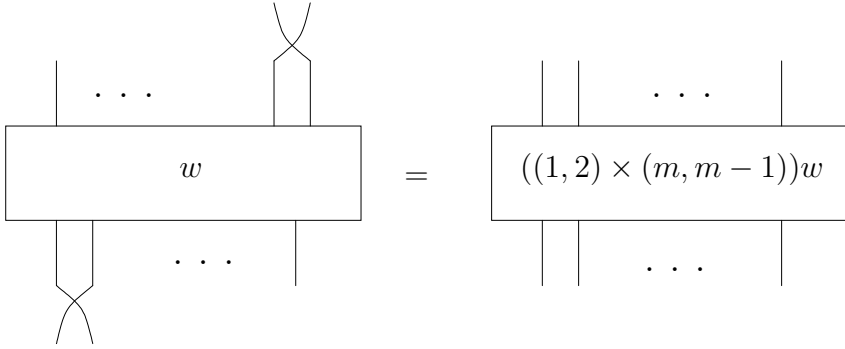


FIGURE 1. Diagrammatic representation of $w \in \mathcal{W}(m)$

eration $\mu_{i,j}$ is diagrammatically represented by joining the i -th output to the j -th input, as in Figure 2. The permutation action is diagrammatically represented by permuting inputs and outputs. For instance, $((1, 2) \times (m-1, m))w$ is represented in Figure 3. Then, the compatibility and associativity conditions for the permutations and contractions say that any combination of operations applied to w depends only on the resulting diagram, up to equivalence (where equivalence means up to homotopy of the legs preserving the left-to-right order of their top endpoints, as well as of their bottom endpoints).

Any wheelspace is, in particular, an \mathbb{S} -module, with diagonal permutation action (σ acts on w by $(\sigma \times \sigma)(w)$). Then, the category of wheelspaces


 FIGURE 2. Diagrammatic representation of $\mu_{i,j}w \in \mathcal{W}(m-1)$

 FIGURE 3. Diagrammatic representation of $((1, 2) \times (m-1, m))w \in \mathcal{W}(m)$

forms a symmetric monoidal category in a canonical way such that the forgetful functor $\text{wheelspaces} \rightarrow \mathbb{S}\text{-mod}$ is symmetric monoidal, and such that, for $w \in \mathcal{W}(m)$, $v \in \mathcal{W}(n)$, and any $i, j \in \{1, \dots, m\}$ and $k, \ell \in \{1, \dots, n\}$,

$$\mu_{i,j}(w \otimes v) = (\mu_{i,j}w) \otimes v, \quad \mu_{k+m, \ell+m}(w \otimes v) = w \otimes (\mu_{k, \ell}v). \quad (3.1.5)$$

Slightly informally, the definition of wheelgebras is then the following:

Definition 3.1.6. A (commutative, associative, Lie, etc.) wheelgebra is a wheelspace which is also a twisted (commutative, associative, Lie, etc.) algebra, such that the algebraic operations (multiplication, bracket, etc.) are morphisms of wheelspaces, i.e., the operations $\mathcal{W}(m_1) \otimes \dots \otimes \mathcal{W}(m_n) \rightarrow \mathcal{W}(m_1 + \dots + m_n)$ are morphisms of $S_{m_1 + \dots + m_n} \times S_{m_1 + \dots + m_n}$ -modules which commute with contractions $\mu_{i,j}, i, j \in \{1, 2, \dots, (m_1 + \dots + m_n)\}$.

For example, an associative wheelgebra \mathcal{W} is a wheelspace equipped with an associative multiplication $\mathcal{W}(m) \otimes \mathcal{W}(n) \rightarrow \mathcal{W}(m+n)$ which is a morphism of $S_{m+n} \times S_{m+n}$ -modules compatible with the contraction.

The word *twisted* above is necessary above since a commutative wheelgebra is not an ordinary commutative algebra, only a twisted commutative algebra (for the associative case, one in fact has an ordinary associative algebra because the braiding does not enter into the associativity axiom). Recall (2.3.4) for the explicit definition of twisted commutative algebra.

A slightly different way to state the above definition is as follows: A commutative wheelgebra is a twisted-commutative algebra together with an extension of the S_m action in degree m to an $S_m \times S_m$ -action such that the original action is obtained by the diagonal embedding $S_m \hookrightarrow S_m \times S_m$, and together with contractions $\mathcal{W}(m) \rightarrow \mathcal{W}(m-1)$ which are compatible with multiplication operations. For any other type of algebra, replace all instances of “commutative” here by some other type of algebra.

For the reader familiar with operads, we can state the above definition more precisely as follows:

Definition 3.1.7. Given any operad \mathcal{O} , an \mathcal{O} -wheelgebra is an algebra over \mathcal{O} in the symmetric monoidal category of wheelspaces.

As always, by an operad we mean a usual \mathbf{k} -linear operad (not an operad in the category of wheelspaces). The above definition then makes sense because the category of wheelspaces is “tensor over vector spaces”: this means that the tensor product $V \otimes W$ of a vector space V and a wheelspace \mathcal{W} is well defined. Precisely, this is given by the symmetric monoidal functor $\mathbf{Vect} \rightarrow \mathbf{Wheelspaces}$ sending a vector space to the corresponding wheelspace concentrated in degree zero. Then, a \mathcal{O} -wheelgebra structure on a wheelspace \mathcal{W} is a map of wheelspaces

$$\bigoplus_{m \geq 0} \mathcal{O}(m) \otimes_{S_m} (W^{\otimes_{\text{wh}} m}) \rightarrow W, \quad (3.1.8)$$

which satisfies the same associativity and unit constraints as in the usual setting. The notation \otimes_{wh} is to remind the reader that this is the tensor product in the category of wheelspaces.

Remark 3.1.9. A commutative wheelgebra is a special case of a (nonunital) wheeled PROP [MMS09]. Namely, a commutative wheelgebra is a nonunital wheeled PROP satisfying the condition that $\text{Hom}(m, n) = 0$ for any $m \neq n$. In the language of commutative wheelgebras, a wheeled PROP unit is an element t in degree one such that $\mu_{1,i}(tx) = x = \mu_{i,1}(tx)$ for all $i \neq 1$. Furthermore, one may define a bigraded version of wheelspaces where, in bidegree (m, n) , one has an action of $S_m \times S_n$. This also forms a symmetric monoidal category, and in this category, commutative algebras are precisely nonunital wheeled PROPs. Then, a unit is as before, but now placed in bidegree $(1, 1)$.

3.2. The commutative wheelgebra $\mathcal{F}(A)$. We now describe our most important example of a wheelgebra, $\mathcal{F}(A)$. We emphasize that the definition is partly motivated by the application given in, e.g., §5.3, and the reader is encouraged to glance there to help understand the reason for it. In one sentence, $\mathcal{F}(A)$ is the commutative wheelgebra freely generated by A in degree 1, subject to the relation that the contraction operation $A \otimes A \rightarrow A$ is the multiplication on A . Explicitly, as a twisted-commutative algebra, it is given by

$$\mathcal{F}(A) := \bigoplus_{m \geq 0} \mathcal{F}_m(A), \quad \mathcal{F}_m(A) := (\text{Ind}_{S_m}^{S_m \times S_m} A^{\otimes m}) \otimes \text{Sym}_{\mathbf{k}} A_{\text{cyc}}, \quad (3.2.1)$$

where the Ind uses the **diagonal embedding** $S_m \hookrightarrow (S_m \times S_m)$, i.e., $\sigma \mapsto (\sigma \times \sigma)$. *Note that we will always use the diagonal embedding in expressions $\text{Ind}_{S_m}^{S_m \times S_m}$ from now on, without mentioning it.* For convenience, let us write the $S_m \times S_m$ -action on $\mathcal{F}_m(A)$ by having the first factor of S_m act on the left and the second act on the right.

In the pictorial representation of Figure 1, the i -th input corresponds to the tensor factor which is changed by the i -th left A -module structure, as follows: Given the element $\sigma_L(f_1 \otimes \cdots \otimes f_m)\sigma_R$, plugging $a \in A$ into the i -th input of the diagram should be thought of as replacing $f_{\sigma_L^{-1}(i)}$ with $af_{\sigma_L^{-1}(i)}$, and similarly plugging a into the i -th output should be thought of as replacing $f_{\sigma_R^{-1}(i)}$ with $af_{\sigma_R^{-1}(i)}$. This interpretation translates the pictorial identities (Figures 2 and 3) into the operations on $\mathcal{F}(A)$ as defined in (3.2.1). We explain these operations precisely as follows.

The $S_m \times S_m$ -module structure in degree m (i.e., on $\mathcal{F}_m(A)$) is immediate from (3.2.1).

The commutative multiplication (let us call it \cdot to distinguish it from $\otimes_{\mathbf{k}}$) is part of the aforementioned twisted-commutative algebra structure, and is given by

$$((\sigma_L X \sigma_R) \otimes Y) \cdot ((\sigma'_L X' \sigma'_R) \otimes Y') = [(\sigma_L \times \sigma'_L)(X \otimes X')(\sigma_R \times \sigma'_R)] \otimes (Y \& Y'), \quad (3.2.2)$$

for $X \in A^{\otimes m}, X' \in A^{\otimes n}$ and any $Y, Y' \in \text{Sym}(A_{\text{cyc}})$.

The contraction operations $\mu_{i,j}$ are obtained by multiplying the i -th output to the j -th input. This means that

$$\begin{aligned} & \mu_{i,j}(a_1 \otimes \cdots \otimes a_n) \\ &= \begin{cases} \sigma_L(a_i a_j \otimes a_1 \otimes \cdots \otimes \hat{a}_i \otimes \cdots \otimes \hat{a}_j \otimes \cdots \otimes a_n) \sigma_R, & \text{if } i \neq j, \\ (a_1 \otimes \cdots \otimes \hat{a}_i \otimes \cdots \otimes a_n) \otimes [a_i]_{\text{cyc}}, & \text{if } i = j, \end{cases} \end{aligned} \quad (3.2.3)$$

where, in the first equation, σ_L and σ_R are the permutations such that the order of the inputs and outputs have the a_k in increasing- k order, except that the term $a_i a_j$ is considered as having i as an input, and as j as an output. The $[a_i]_{\text{cyc}}$ in the second line means the image of a_i in A_{cyc} . Finally, the

hats (\hat{a}_i and \hat{a}_j) mean that the given terms (a_i and a_j) are **omitted** from the tensor product.

The above formulas determine all of the contraction action by compatibility with permutations (3.1.3), linearity, and the condition $\mu_{i,j}(X \otimes Y) = \mu_{i,j}(X) \otimes Y$ if $Y \in \text{Sym } A_{\text{cyc}}$.

Remark 3.2.4. The commutative wheelgebra $\mathcal{F}(A)$ is not merely a nonunital wheeled PROP (cf. Remark 3.1.9), but in fact is unital, with unit $1 \in A \subset \mathcal{F}_1(A)$.

Remark 3.2.5. In [MMS09], the notion of algebra over a wheeled PROP was defined (which generalizes the notion of algebra over a PROP, and of algebra over an operad). Since, as in Remark 3.1.9, a commutative wheelgebra is a special kind of wheeled PROP, one obtains a notion of algebra over a commutative wheelgebra. In the case $\mathcal{W} = \mathcal{F}(A)$, finite-dimensional algebras S over \mathcal{W} are the same as *representations of A* with underlying vector space S (since the compatibility with contractions says that $\rho([a]_{\text{cyc}}) = \text{tr} \rho(a)$). If we generalize S from a finite-dimensional vector space to a finitely-generated free module over a commutative \mathbf{k} -algebra B , we get families of representations parametrized by B . This theory (and its application to polyvector fields, differential operators, BV structures, and so on) will be discussed in §6.

3.3. Differential operators on commutative wheelgebras. We may now invoke the categorical definition of differential operators:

Definition 3.3.1. For any commutative wheelgebra \mathcal{W} , let $\mathcal{D}(\mathcal{W})$ denote the almost-commutative wheelgebra of differential operators on \mathcal{W} .

As before, the associated graded of an almost-commutative wheelgebra is a Poisson wheelgebra.

Explicitly, a differential operator of twisted-degree i on a commutative wheelgebra \mathcal{W} is a differential operator D of twisted-degree i on the underlying twisted-commutative algebra of \mathcal{W} , which satisfies the following condition:

$$D \circ \mu_{j,k} = \mu_{i+j,i+k} \circ D. \quad (3.3.2)$$

The contractions of $\mathcal{D}(\mathcal{W})$ are given by the equation

$$\mu_{i,j}(D) = \mu_{i,j} \circ D. \quad (3.3.3)$$

Example 3.3.4. For any associative algebra A , one may consider the wheelgebra $\mathcal{D}(\mathcal{F}(A))$, which we will sometimes call the *wheeled differential operators* on A .

Example 3.3.5. The differential operators $\mathcal{D}(\mathcal{F}(T_{\mathbf{k}}V))$ include, as a subalgebra, the operators $\mathcal{D}(T_{\mathbf{k}}V)$: the latter are exactly the differential operators which preserve the **non-wheeled** subalgebra of $\mathcal{F}(T_{\mathbf{k}}V)$ generated by V (this subalgebra is isomorphic as a twisted algebra to $T_{\mathbf{k}}V$). Note also that all elements of $\mathcal{D}(\mathcal{F}(T_{\mathbf{k}}V))$ are determined by their restriction to the aforementioned subalgebra.

3.4. Lie and Poisson wheelgebras. A Lie or Poisson wheelgebra is a Lie or Poisson algebra in the category of wheelspaces, respectively.

Explicitly, a Lie wheelgebra bracket on a wheelspace \mathcal{W} is a twisted Lie bracket $\{-, -\} : W \otimes W \rightarrow W$ satisfying

$$\{\mu_{i,j}a, b\} = \mu_{i,j}\{a, b\}, \quad 1 \leq i, j \leq |a|. \quad (3.4.1)$$

A Poisson wheelgebra is a commutative wheelgebra equipped with a Lie wheelgebra bracket, which satisfies the Leibniz rule. The Leibniz rule may equivalently be stated as saying that the underlying twisted structures form a twisted Poisson algebra.

We remark that there is a construction of universal enveloping wheelgebra of Lie wheelgebras and a PBW theorem for Lie wheelgebras, which is completely analogous to the usual case (for the twisted, rather than wheeled, setting, this has been known for a long time in topology; see, e.g., [Bar78]).

Similarly, one may construct a Koszul complex for a Lie wheelgebra (giving a resolution of the “augmentation” module of the universal enveloping wheelgebra). Specifically, the analogue of $\text{Sym } \mathfrak{g}$ is $\text{Sym}_{\text{wh}}(L)$, where for an arbitrary wheelspace L , $\text{Sym}_{\text{wh}}(L)$ is defined to be the free commutative wheelgebra generated by L (this means, the commutative wheelgebra satisfying the universal property that $L \hookrightarrow \text{Sym}_{\text{wh}}(L)$ is an embedding of wheelspaces; this can also be presented explicitly with some work). Similarly, the analogue of $\Lambda \mathfrak{g}$ is the *super* version $\text{SuperSym}_{\text{wh}}(L(1))$, where the braiding is by super-braidings τ_σ with respect to the grading $|L(1)| = 1$, and uses and one uses supercommutators rather than commutators. The result is a supercommutative wheelgebra. Then, if L is a Lie wheelgebra, $\text{SuperSym}_{\text{wh}}(L \oplus L(1))$ (with $|L| = 0, |L(1)| = 1$) may be equipped with a Koszul differential, summing with sign over applying the bracket, for $f \in \text{Sym}_{\text{wh}}(L)$ and $x_1, \dots, x_m \in L(1)$:

$$\begin{aligned} & d(f \otimes (x_1 \otimes \cdots \otimes x_m)) \\ &= \sum_{i=1}^m (-1)^{i+1} (\text{Id}_{S_{|f|}} \times \sigma_i) (f \cdot x_i \otimes (x_1 \otimes \cdots \otimes \hat{x}_i \cdots \otimes x_m)) (\text{Id}_{S_{|f|}} \times \sigma_i) \\ & - \sum_{i < j} (-1)^{i+j} (\text{Id}_{S_{|f|}} \times \sigma_{ij}) (f \otimes (\{x_i, x_j\} \otimes x_1 \otimes \cdots \otimes \hat{x}_i \cdots \otimes \hat{x}_j \cdots \otimes x_m)) (\text{Id}_{S_{|f|}} \times \sigma_{ij}), \end{aligned} \quad (3.4.2)$$

where σ_i is the inverse of the permutation sending $x_1 \otimes \cdots \otimes x_m$ to $x_i \otimes x_1 \otimes \cdots \otimes \hat{x}_i \cdots \otimes x_m$ and σ_{ij} is the inverse of the permutation that sends $x_1 \otimes \cdots \otimes x_m$ to $x_i \otimes x_j \otimes x_1 \otimes \cdots \otimes \hat{x}_i \cdots \otimes \hat{x}_j \cdots \otimes x_m$.

Details will be discussed elsewhere.

3.5. Wheeled Poisson structure on the double cotangent bundle. We briefly recall the *even* version of Van den Bergh’s double Schouten-Nijenhuis bracket [VdB08] (we will explain things slightly differently from [VdB08]).

An initial idea is to try to produce an analogue for $\mathbb{D}\text{er}(A)$ of the standard Lie bracket of vector fields. Naively, this involves a formula of the form $[\xi, \eta] = \xi \circ \eta - \eta \circ \xi$. (One certainly wants a commutator-type formula if this

is to come from the associated graded of a space of differential operators.) Of course, as written, this cannot work, since $\eta(A) \subset A \otimes A$, and so after applying η , we cannot apply ξ : we have to choose either $\text{Id} \otimes \xi$ or $\xi \otimes \text{Id}$. So, there are four ways to apply one of ξ, η and then the other. Our goal is to obtain another derivation. It turns out that there is a two-dimensional vector space of linear combinations of such compositions that always yield derivations $A \rightarrow (A \otimes A \otimes A)$, using the *outer* bimodule action on $A \otimes A \otimes A$. This vector space is spanned by

$$(\xi \otimes \text{Id}) \circ \eta - (\text{Id} \otimes \eta) \circ \xi, \quad (\text{Id} \otimes \xi) \circ \eta - (\eta \otimes \text{Id}) \circ \xi. \quad (3.5.1)$$

These two elements form a *canonical* basis, up to scaling both by the same nonzero constant, as will be explained after the following definitions.

Definition 3.5.2 ([VdB08]). Define the brackets

$$\{\{-, -\}_L : \text{Der}(A) \otimes \text{Der}(A) \rightarrow \text{Der}(A) \otimes A, \quad (3.5.3)$$

$$\{\{-, -\}_R : \text{Der}(A) \otimes \text{Der}(A) \rightarrow A \otimes \text{Der}(A) \quad (3.5.4)$$

to be the unique elements such that the associated derivations $A \rightarrow A \otimes A \otimes A$ (viewing $A \otimes A \otimes A$ as an A -bimodule using the **outer** action) are

$$\{\{\xi, \eta\}_L^\sim = (\xi \otimes \text{Id}) \circ \eta - (\text{Id} \otimes \eta) \circ \xi, \quad \{\{\xi, \eta\}_R^\sim = (\text{Id} \otimes \xi) \circ \eta - (\eta \otimes \text{Id}) \circ \xi. \quad (3.5.5)$$

Definition 3.5.6 ([VdB08]). A double Poisson algebra is an associative algebra A together with a \mathbf{k} -bilinear bracket $\{\{-, -\} : A \otimes A \rightarrow A \otimes A$ satisfying the conditions

$$\{\{\xi, \eta_1 \eta_2\} = \{\{\xi, \eta_1\} \eta_2 + \eta_1 \{\{\xi, \eta_2\}\}, \quad (3.5.7)$$

$$\{\{-, -\} = -\tau_{(21)} \circ \{\{-, -\} \circ \tau_{(21)}, \quad (3.5.8)$$

$$\{\{\xi, a\eta\} = a\{\{\xi, \eta\} + \{\{\xi, a\}\}\eta, \quad (3.5.9)$$

$$\sum_{i=0}^2 \tau_{(123)^i} \circ (\{\{-, -\}^{23} \circ \{\{-, -\}^{12}) \circ \tau_{(123)^{-i}} = 0. \quad (3.5.10)$$

Remark 3.5.11. Note that the resulting map $A \otimes A \rightarrow A \otimes A$ is a derivation from the first component to the *inner* action on the target, and a derivation from the second component to the *outer* action on the target. In some derivation sense, this is saying that *left-multiplication components are reversed*: multiplying on the left of the first component on the domain $A \otimes A$ is sent to multiplying the left of the second component of $A \otimes A$ in the image, up to a derivation term. More precisely, for such a derivation D ,

$$D(ab_1 \otimes b_2) = (1 \otimes a)D(b_1 \otimes b_2) + D(a \otimes b_2)(b_1 \otimes 1), \quad (3.5.12)$$

and saying that “left-multiplication components are reversed” is saying that, if we compare the LHS with the first term on the RHS, the a changes from the first to the second component. On the other hand, *right-multiplication components are preserved*, in the sense that, comparing the LHS with the second term on the RHS, the b_1 remains in the first component. We interpret

this as saying that *the double bracket prefers right-multiplication*. We will eliminate this preference when we introduce the wheeled Poisson version of the above. \diamond

Following Van den Bergh, we introduce a double bracket, which is an *even* version of the *odd* double bracket, the Schouten-Nijenhuis double bracket, considered in [VdB08].

Definition 3.5.13. [VdB08] Define $\{\!\{-, -\}\!\} : (T_A\mathbb{D}\mathrm{er}(A))^{\otimes 2} \rightarrow (T_A\mathbb{D}\mathrm{er}(A))^{\otimes 2}$ to be the unique double Poisson bracket such that

$$\{\!\{a, b\}\!\} = 0 \quad \text{if } a, b \in A; \quad (3.5.14)$$

$$\{\!\{\xi, b\}\!\} = \xi(b) \quad \text{if } \xi \in \mathbb{D}\mathrm{er}(A), b \in A; \quad (3.5.15)$$

$$\{\!\{\xi, \eta\}\!\} = \{\!\{\xi, \eta\}\!\}_L + \{\!\{\xi, \eta\}\!\}_R, \quad \text{if } \phi, \psi \in \mathbb{D}\mathrm{er}(A). \quad (3.5.16)$$

The above double bracket $\{\!\{-, -\}\!\}$ is the *unique* double Poisson bracket, up to a constant factor, satisfying (3.5.14), (3.5.15) such that $\{\!\{\xi, \eta\}\!\}$ is given by a fixed linear combination of elements (3.5.1) written to lie in either $A \otimes \mathbb{D}\mathrm{er}(A)$ or $\mathbb{D}\mathrm{er}(A) \otimes A$, and such that this works for *any* algebra A . This explains why the elements in (3.5.1) form a canonical basis up to scaling (we must have the first map to $\mathbb{D}\mathrm{er}(A) \otimes A$ and the second map to $A \otimes \mathbb{D}\mathrm{er}(A)$ so as to preserve the double Leibniz rule (3.5.7)).

In this article, we will use a modification of Van den Bergh's double bracket above. There are two motivations. One is to correct the "preference" that the double bracket makes for preserving right-multiplication components (Remark 3.5.11), by allowing one to keep track of *both* left- and right-multiplication components. The other motivation is to incorporate this into the general *wheelgebra* framework that allows one to relate to differential operators, (almost-)commutative structures, etc.

Definition 3.5.17. Let the **wheeled Poisson bracket** on $\mathcal{F}(T_A\mathbb{D}\mathrm{er}(A))$ be the unique one such that, for all $f, g \in T_A\mathbb{D}\mathrm{er}(A) \subset \mathcal{F}_1(T_A\mathbb{D}\mathrm{er}(A))$,

$$\{f, g\} = (12)\{\!\{f, g\}\!\}. \quad (3.5.18)$$

Here, $\{\!\{-, -\}\!\} : T_A\mathbb{D}\mathrm{er}(A) \otimes T_A\mathbb{D}\mathrm{er}(A) \rightarrow T_A\mathbb{D}\mathrm{er}(A) \otimes T_A\mathbb{D}\mathrm{er}(A)$ is the double Poisson bracket of Definition 3.5.13.

Using the above definition, it is now true that the wheeled Poisson bracket on $\mathcal{F}(T_A\mathbb{D}\mathrm{er}(A))$ transforms the i -th left-multiplication on $(f \otimes g)$, i.e., left-composing $(f \otimes g)$ by $[\mathrm{Id} \otimes (1^{\otimes(i-1)} \otimes a \otimes 1^{\otimes n-i}) \otimes \mathrm{Id}]$ for $a \in A$, to the i -th left-multiplication on the element $\{f, g\}$ plus an appropriate derivation term (i.e., a term obtained from (1) applying g to a as a sum over the derivations in $\mathbb{D}\mathrm{er}(A)$ appearing in g , (2) tensoring by f , and (3) applying a sign and a permutation of components); similarly for right-multiplication. This corrects the double Poisson phenomenon of Remark 3.5.11, and preserves additional information.

Remark 3.5.19. It is *not* true that a wheeled Poisson bracket on $\mathcal{F}(A)$ is equivalent to a double bracket on A . Wheeled Poisson brackets are a generalization of double Poisson brackets. Precisely, the former corresponds to wheeled Poisson brackets on $\mathcal{F}(A)$ of the form $\{a, b\} \in (12)(A \otimes A) \subset \mathcal{F}_2(A)$ for all $a, b \in A \subset \mathcal{F}_1(A)$. In the smooth case, in the description of Theorem 3.6.7, wheeled Poisson brackets are given by elements of $\mathcal{F}_2(T_A \mathbb{D}er(A))[2]$ which have zero Schouten-Nijenhuis bracket with themselves, while double Poisson brackets are given by the subset of such elements in $(T_A \mathbb{D}er(A))_{\text{cyc}}[2]$ (this description of double Poisson brackets was given in [VdB08]).

3.6. Lie wheelgebroids and structure of $\mathcal{D}(\mathcal{F}(A))$. We are interested in the part of $\mathcal{F}(T_A \mathbb{D}er(A))$ that involves only one copy of $\mathbb{D}er(A)$. Thus, we consider the space

$$\mathcal{W}\mathbb{D}er(A) := \mathcal{F}_{\geq 1}(A) \otimes_{A^e} \mathbb{D}er(A) \cong \mathcal{F}(T_A \mathbb{D}er(A))[1], \quad (3.6.1)$$

where the $[1]$ means degree-one with respect to $|\mathbb{D}er(A)| = 1$, $|A| = 0$. Here, $\mathcal{F}_{\geq 1}(A) \otimes_{A^e} \mathbb{D}er(A)$ means the span of elements $a \otimes_{A^e} \xi$, for $a \in \mathcal{F}_{\geq 1}(A)$ and $\xi \in \mathbb{D}er(A)$; the multiplication \otimes_{A^e} can also be expressed as first taking \otimes and then performing the two appropriate contractions (here, $\mu_{1,|a|} \circ \mu_{|a|+1,1}$).

It is easy to see that the constructions of the previous section restrict to the statement that $\mathcal{W}\mathbb{D}er(A)$ is a Lie wheelgebra (this can also be obtained without using $T_A \mathbb{D}er(A)$ at all: one needs only $\{\xi, \eta\}$ for $\xi, \eta \in \mathbb{D}er(A)$, and the action of $\mathbb{D}er(A)$ on A). Furthermore, $\mathcal{F}(T_A \mathbb{D}er(A))$ may be interpreted as $\text{Sym}_{\mathcal{F}(A)} \mathcal{W}\mathbb{D}er(A)$; one then has the wheeled analogue of the usual result that this construction produces a Poisson algebra.

Also, there is a construction of universal enveloping wheelgebra of a Lie wheelgebra (or wheelgebroid), and a “wheeled PBW theorem” which says that

$$\text{gr } U_{\mathcal{F}(A)} \mathcal{W}\mathbb{D}er(A) \cong \text{Sym}_{\mathcal{F}(A)} \mathcal{W}\mathbb{D}er(A).$$

This is another way to obtain the same wheeled Poisson bracket. Moreover, in the case that A is smooth, $\mathcal{D}(\mathcal{F}(A)) \cong U_{\mathcal{F}(A)} \mathcal{W}\mathbb{D}er(A)$ (see Theorem 3.6.7 below). This is the analogue of the classical statement that differential operators on a manifold are the universal enveloping algebra of the Lie algebroid of vector fields over the space of functions. To make these statements precise, we must interpret $\mathcal{W}\mathbb{D}er(A)$ not merely as a Lie wheelgebra, but as a Lie wheelgebroid over $\mathcal{F}(A)$:

Definition 3.6.2. A wheelmodule \mathcal{M} over a wheelgebra \mathcal{W} is a module over \mathcal{W} in the category of wheelspaces.

Explicitly, a wheelmodule \mathcal{M} over a wheelgebra \mathcal{W} is a wheelspace together with a \mathbf{k} -linear map $\rho : \mathcal{W} \otimes \mathcal{M} \rightarrow \mathcal{W}$ such that

1. $\mathcal{W}(m) \otimes \mathcal{M}(n) \rightarrow \mathcal{W}(m+n)$ is a morphism of $S_{m+n} \times S_{m+n}$ -modules;
2. $\rho((w_1 \otimes w_2) \otimes x) = \rho(w_1 \otimes \rho(w_2 \otimes x))$ (for $w_1, w_2 \in \mathcal{W}, x \in \mathcal{M}$,
3. $\rho(\mu_{i,j}(w) \otimes x) = \mu_{i,j} \rho(w \otimes x)$, $\rho(w \otimes \mu_{i,j}(x)) = \mu_{i+|w|,j+|w|} \rho(w \otimes x)$ (for $w \in \mathcal{W}, x \in \mathcal{M}$).

When there is no confusion, we will write $wx := \rho(w \otimes x)$.

A *Lie wheelgebroid* is a Lie algebroid in the category of wheelspaces. We also give this definition explicitly, since the categorical definition of Lie algebroids is probably less familiar.

Definition 3.6.3. A *Lie wheelgebroid* L over a commutative wheelgebra \mathcal{W} is a wheelmodule over \mathcal{W} together with

1. an action $\theta : L \otimes W \rightarrow W$, which satisfies (denoting $\theta_x(w) := \theta(x, w)$)

$$w_1 \theta_x(w_2) = \theta_{w_1 x}(w_2), \quad (3.6.4)$$

$$\theta_x(w_1 w_2) = \theta_x(w_1) w_2 + (\sigma \times \text{Id}) w_1 \theta_x(w_2), \quad (3.6.5)$$

where $\sigma \in S_{|w_1|+|x|}$ is the permutation which sends $1, \dots, |w_1|$ to $|x| + 1, \dots, |x| + |w_1|$, and is increasing on $1, \dots, |w_1|$ and on $|w_1| + 1, \dots, |w_1| + |x|$;

2. a Lie bracket in wheelspaces, $\{-, -\} : L \otimes L \rightarrow L$, satisfying the Leibniz rule

$$\{wx, y\} = (w\{x, y\}) + \omega(\theta_y(w)x), \quad w \in W, \quad x, y \in L, \quad (3.6.6)$$

where $\omega \in S_{|w|+|x|+|y|}$ is the permutation which sends $1, \dots, |y|$ to the end ($|x| + |w| + 1, \dots, |x| + |w| + |y|$), and is increasing on $1, \dots, |y|$ and on $|y| + 1, \dots, |y| + |x| + |w|$.

It is easy to see that $\mathcal{W}\text{Der}(A)$ is a Lie wheelgebroid over $\mathcal{F}(A)$. Explicitly, the action of $\mathcal{F}(A)$ on $\mathcal{W}\text{Der}(A)$ is the tensor product; the action of $\mathcal{W}\text{Der}(A)$ on $\mathcal{F}(A)$ is by applying the element of $\text{Der}(A)$ and multiplying the rest accordingly. The Lie structure on $\mathcal{W}\text{Der}(A)$ is the wheeled Poisson bracket. Namely, for two elements $\xi, \eta \in \text{Der}(A)$, we apply $(\xi \otimes \text{Id}) \circ \eta - (\text{Id} \otimes \eta) \circ \xi$ and similarly the other way, to get a new element of $(\text{Der}(A) \otimes A) \oplus (A \otimes \text{Der}(A))$; to generalize to bracketing two elements of $\mathcal{W}\text{Der}(A)$ involves also summing over applying ξ to copies of A or $A/[A, A]$, and similarly for η .

For any Lie wheelgebroid L over a commutative wheelgebra \mathcal{W} , one may define the universal enveloping wheelgebra $U_W L$ in a straightforward way. We remark also that the Koszul complex construction of Section 3.4 can be done for Lie wheelgebroids.

Theorem 3.6.7. *For any smooth associative algebra A , we have*

- (i) *The wheelgebra $\mathcal{D}(\mathcal{F}(A))$ is canonically isomorphic to the universal enveloping wheelgebra of the Lie wheelgebroid $\mathcal{W}\text{Der}(A)$.*
- (ii) *The Poisson wheelgebra $\text{gr } \mathcal{D}(\mathcal{F}(A))$ is isomorphic to $\mathcal{F}(T_A \text{Der}(A))$, equipped with the wheeled Poisson bracket.*

We will only prove (ii). It is not difficult to deduce (i) from this.

Note that, if the algebra A is not smooth, we still have a natural map of wheeled Poisson algebras $\mathcal{F}(T_A \text{Der}(A)) \rightarrow \text{gr } \mathcal{D}(\mathcal{F}(A))$. The reason is that $\text{Der}(A)$ and A always act on $\mathcal{F}(A)$, regardless of whether A is smooth. For

example, any derivation $\theta \in \mathbb{D}\mathrm{er}(A)$ acts by

$$\theta(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \sum_i \sigma_i(a_1 \otimes a_2 \otimes \cdots \otimes a_{i-1} \otimes \theta(a_i) \otimes a_{i+1} \otimes \cdots \otimes a_n), \quad (3.6.8)$$

where $\sigma_i \in S_{n+1}$ is an appropriate permutation. The action of A is by tensoring on the left. This action extends to $T_{\mathbf{k}}(\mathbb{D}\mathrm{er}(A) \oplus A)$ by replacing tensor product of elements by composition of the corresponding operators. One may construct from this a \mathbf{k} -linear map $\mathcal{F}(T_A \mathbb{D}\mathrm{er}(A)) \rightarrow \mathcal{D}(\mathcal{F}(A))$ (using twisted symmetrization, which in fact yields a map of \mathbb{S} -modules), which induces a map $\mathcal{F}(T_A \mathbb{D}\mathrm{er}(A)) \rightarrow \mathrm{gr} \mathcal{D}(\mathcal{F}(A))$ of commutative wheelgebras. However, the map need not be injective or surjective. Similarly, we have a natural map of wheelgebras $U_{\mathcal{F}(A)} \mathcal{W}\mathrm{Der}(A) \rightarrow \mathcal{D}(\mathcal{F}(A))$, which need not be injective or surjective.

Before proving the theorem, we need to define the notion of principal symbol.

Definition 3.6.9. Define the *n-th principal symbol* of a differential operator $D \in \mathcal{D}(B)_{\leq n}$ on a *twisted-commutative algebra* (or commutative algebra in any appropriate symmetric monoidal category) B to be a map $\Gamma_n(D) : B^{\otimes n} \rightarrow B$, given by $b_1 \otimes \cdots \otimes b_n \mapsto [[[D, b_1], b_2], \dots, b_n](1)$. The term “principal symbol” of a differential operator D refers to $\Gamma_n(D)$ where D has order $\leq n$ but not order $\leq n-1$.

One can prove by a straightforward computation that the principal symbol is given by the following explicit formula (for the twisted case):

$$\Gamma_n(D)(b_1 \otimes \cdots \otimes b_n) = \sum_{I \subset \{1, \dots, n\}} (-1)^{|I|} \cdot (\mathrm{Id}_{S_{|D|}} \times \sigma_I) \circ D\left(\prod_{i \in I} b_i\right) \prod_{j \notin I} b_j, \quad (3.6.10)$$

where $\sigma_I \in S_n$ is the permutation which, if considered to act by permuting the indices, would rearrange them in order $1, \dots, n$ (i.e., σ_I sends the ordered set $I + I^c$ to $(1, 2, \dots, n)$, where I^c is the complement of I , $+$ is concatenation, and I, I^c are assumed to have the increasing order). Also, the products above are in increasing order of indices from left to right. In the arbitrary categorical setting, we replace the permutation $\mathrm{Id}_{S_{|D|}} \times \sigma_I$ by the corresponding composition of braidings.

Note that the *n*-th principal symbol $\Gamma_n(D)$ is zero if and only if D is a differential operator of order $\leq n-1$.

Proof of Theorem 3.6.7. We only prove part (ii); part (i) easily follows. Recall that $\mathcal{D}(A)_{\leq n}$ denotes the subspace of operators of order $\leq n$. We may consider the map

$$\mathcal{D}(A)_{\leq n} \xrightarrow{\Gamma_n} \mathrm{Hom}_{\mathbf{k}}(\mathcal{F}(A)^{\otimes n}, \mathcal{F}(A)), \quad (3.6.11)$$

whose kernel is clearly the differential operators of order $\leq n-1$, and hence induces an embedding $\mathrm{gr}_n \mathcal{D}(A) \hookrightarrow \mathrm{Hom}_{\mathbf{k}}(\mathcal{F}(A)^{\otimes n}, \mathcal{F}(A))$. Let us study the

image. It is straightforward to check that the image is a derivation in all tensor components, in the sense that (for homogeneous $a_1, \dots, a_n, a'_i \in \mathcal{F}(A)$)

$$\begin{aligned} \Gamma_n(D)(a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i a'_i \otimes a_{i+1} \otimes \cdots \otimes a_n) &= \mu_{|D|+i, |D|+(i+1)}(\text{Id}_{S_{|D|}} \times \sigma_1) \cdot \\ &\quad (\Gamma_n(D) \cdot (a_1 \otimes \cdots \otimes a_{i-1} \otimes a'_i \otimes a_{i+1} \otimes \cdots \otimes a_n) \otimes a_i)(\text{Id}_{S_{|D|}} \times \sigma_1) \\ &+ \mu_{|D|+i, |D|+(i+1)}(\text{Id} \times \sigma_2)(\Gamma_n(D)(a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n) \otimes a'_i)(\text{Id} \times \sigma_2), \end{aligned} \quad (3.6.12)$$

where $\sigma_1, \sigma_2 \in S_{|a_1|+\dots+|a_n|+|a'_i|}$ are the respective permutations which would rearrange the a symbols back to the order $a_1, \dots, a_i, a'_i, a_{i+1}, \dots, a_n$.

The image is determined by its restriction to $\text{Hom}_{\mathbf{k}}(A^{\otimes n}, \mathcal{F}_{\geq n}(A))$, in view of the compatibility of contraction maps on the domain and the image. Using the rightmost $(A^e)^{\otimes n}$ -bimodule structure on $\mathcal{F}_{\geq n}(A)$, we see that these restrictions land in

$$\begin{aligned} \text{Der}(A^{\otimes n}, \mathcal{F}_{\geq n}(A)) &= \text{Der}(A, \text{Der}(A^{\otimes(n-1)}, \mathcal{F}_{\geq n}(A))) \\ &= \mathbb{D}\text{er}(A) \otimes_{A^e} \text{Der}(A^{\otimes(n-1)}, \mathcal{F}_{\geq n}(A)), \end{aligned} \quad (3.6.13)$$

where the last isomorphism is due to the smoothness of A . This way, by induction, we get $\text{Der}(A^{\otimes n}, \mathcal{F}_{\geq n}(A)) \cong \mathbb{D}\text{er}(A)^{\otimes n} \otimes_{(A^e)^{\otimes n}} \mathcal{F}_{\geq n}(A)$. Thus, we obtain an embedding

$$\text{gr}_n \mathcal{D}(A) \hookrightarrow (\mathbb{D}\text{er}(A)^{\otimes n}) \otimes_{(A^e)^{\otimes n}} (\mathcal{F}_{\geq n}(A)). \quad (3.6.14)$$

It remains to show that this map is surjective. To see this, we may define, for any $\theta_1, \dots, \theta_n \in \mathbb{D}\text{er}(A)$, and any $X \in \mathcal{F}_{\geq n}(A)$, a differential operator in $\mathcal{D}(\mathcal{F}(A))$ mapping to $(\theta_1 \otimes \cdots \otimes \theta_n) \otimes_{(A^e)^{\otimes n}} X$, as follows. For any i , we may consider the operator $\theta_i : \mathcal{F}_{\bullet}(A) \rightarrow \mathcal{F}_{\bullet+1}(A)$, given by

$$\begin{aligned} \theta_i(a_1 \otimes \cdots \otimes a_m) &= \sum_{j=1}^m (1, j)(\theta_i(a_j)' \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{j-1} \otimes \theta_i(a_j)'' \otimes a_{j+1} \otimes \cdots \otimes a_m), \end{aligned} \quad (3.6.15)$$

extended so as to be an S_m -module map $\mathcal{F}_m(A) \rightarrow \mathcal{F}_{m+1}(A)$ (viewing $\mathcal{F}_{m+1}(A)$ as an S_m -module by the composition map $S_m \rightrightarrows S_m \times S_1 \hookrightarrow S_{m+1}$), and so as to be compatible with contractions.

Then, for any $X \in \mathcal{F}_p(A)$ with $p \geq n$, we define a differential operator D mapping to $(\theta_1 \otimes \cdots \otimes \theta_n) \otimes_{(A^e)^{\otimes n}} X$ as follows: for any $Y \in \mathcal{F}_m(A)$, first consider $Y' = \theta_1 \circ \theta_2 \circ \cdots \circ \theta_n(Y) \in \mathcal{F}_{m+n}(A)$. Then, we let $D(Y) := X \otimes_{(A^e)^{\otimes n}} Y' \in \mathcal{F}_{\geq m}(A)$, considering Y' as an element of an $(A^e)^{\otimes n}$ -module via the first n components of the $(A^e)^{\otimes(n+m)}$ -action, and similarly X via the first n components of the $(A^e)^{\otimes(n+m)}$ -action. The result $D(Y) \in \mathcal{F}_{(p-n)+m}(A)$ has $(A^e)^{\otimes(p-n)+m}$ -action given by: first the remaining $A^{\otimes(p-n)}$ -action on X , and then the remaining $A^{\otimes m}$ -action on Y' . Also, here, tensoring over $(A^e)^{\otimes n}$ is the same as taking a tensor product over \mathbf{k} and applying the appropriate $2n$ contraction maps.

It is not difficult to show that the above D indeed maps to $(\theta_1 \otimes \cdots \otimes \theta_n) \otimes_{(A^e)^{\otimes n}} X$ under the principal symbol map. Furthermore, it follows from the construction that the obtained identification is compatible with permutations and contractions (for example, applying such operations to the last $p - n$ components of X above is the same as applying these operations to the first $p - n$ components of the result).

It remains to consider the Poisson structure. It suffices to show that the induced wheeled Poisson bracket agrees with the double Poisson bracket from [VdB08] when restricted to $T_A \mathbb{D}er(A) \otimes T_A \mathbb{D}er(A)$ (it is clear that the restriction to $A \otimes A$ determines the bracket). In fact, it is enough to check the restrictions to $A \otimes A$, $\mathbb{D}er(A) \otimes A$, and $\mathbb{D}er(A) \otimes \mathbb{D}er(A)$. The first restriction is clearly zero, for degree reasons. We have to show that, if $\xi, \eta \in \mathbb{D}er(A)$, and $\phi, \psi \in \mathcal{D}(A)_{\leq 1}$ satisfy $\Gamma_1(\phi) = \xi, \Gamma_1(\psi) = \eta$, then

$$[\phi, a] = \xi(a), \quad \Gamma_1[\phi, \psi] = \{\xi, \eta\}, \quad \forall a \in A. \quad (3.6.16)$$

The first identity is immediate from the definition of Γ_1 . For the second, we recall the definition of $\{\{-, -\}\}$ and $\{-, -\}$. This says that $\{\xi, \eta\}$, viewed as a map $A \rightarrow \text{Ind}_{S_3}^{S_3 \times S_3}(A \otimes A \otimes A)$ (sending the input and output for A in the domain to the third input and third output in the image), sums over all ways to apply η first and then ξ . Since we may choose ϕ and ψ to effectively sum over applying ξ and η , respectively, to all A components that appear, this proves the desired equality. \square

4. Torsion of bimodule connections on $\mathbb{D}er(A)$ and $\Omega^1 A$

4.1. Sign conventions. Unless otherwise specified, we will work in the super $\mathbb{Z} \oplus \mathbb{Z}$ -graded context, with bidegrees $|A| = (0, 0)$, $|\mathbb{D}er(A)| = (1, 0)$, $|\Omega^1 A| = (0, -1)$.

Then, we use the corresponding superbraiding,

$$\tau_{(21)}(M \otimes N) := s(|M|, |N|)(N \otimes M), \quad (4.1.1)$$

where $s((a, b), (c, d)) = (-1)^{ac+bd}$.

The grading $|A|$ used in the definition of τ above will be called the “ τ -grading” (to distinguish from other gradings that will arise, e.g., \mathbb{S} -module gradings).

Notation 4.1.2. Let $\bar{\tau}_\sigma$ be the non-super version of Notation 1.7.1: $\bar{\tau}_{(21)}(A \otimes B) = B \otimes A$ for all A, B .

4.2. Connections on a bimodule. This subsection is a reminder of basic facts and definitions from [CQ95]; accordingly, we omit the citations of [CQ95]. Let A be an associative algebra over \mathbf{k} .

Definition 4.2.1. Let M be any A -bimodule. Then a left connection ∇_ℓ on M is a right A -module map $\nabla_\ell : M \rightarrow \Omega^1 A \otimes_A M$ satisfying $\nabla_\ell(am) = a\nabla_\ell(m) + da \otimes m$ for any $m \in M, a \in A$. Similarly, a right connection ∇_r on M is a left A -module map $\nabla_r : m \rightarrow M \otimes_A \Omega^1 A$ satisfying $\nabla_r(ma) = \nabla_r(m)a + m \otimes da$.

Recall that being a right A -module map means $\nabla_\ell(ma) = \nabla_\ell(m)a$.

Definition 4.2.2. A connection $\nabla = (\nabla_\ell, \nabla_r)$ on an A -bimodule M is a collection of a left connection ∇_ℓ and a right connection ∇_r .

Remark 4.2.3. There is also a similar notion of connection on a left or right A -module. However, a connection on an A -bimodule is **not** the same as a connection on the corresponding (left or right) A^e -module: the latter is a finer notion. See [CQ95, §8].

Proposition 4.2.4. *A connection on an A -bimodule M exists if and only if M is a projective A^e -module.*

Definition 4.2.5. For each $n \geq 1$, let $\Omega^n A := (\Omega^1 A)^{\otimes_{A^e} n}$, $\Omega^0 A := A$, and $\Omega A := \bigoplus_{n \geq 0} \Omega^n A$. If $\eta \in \Omega^n A \subset \Omega A$, we write $|\eta| = n$ and say η has degree n .

In the proposition below, $|\eta| = n$ means $\eta \in M \otimes_A \Omega^n A$ (or $\Omega^n A \otimes_A M$ for that matter).

Proposition 4.2.6. *Any left connection on M extends uniquely to an operator of degree one on $\Omega A \otimes_A M$ by the condition $\nabla(\omega\eta) = d(\omega)\eta + (-1)^{|\omega|}\omega\nabla(\eta)$. Similarly, any right connection on M extends uniquely to $M \otimes_A \Omega A$ by $d(\eta\omega) = \nabla(\eta)\omega + (-1)^{|\eta|}\eta d(\omega)$.*

4.3. Torsion of a connection on $\mathbb{D}er(A)$. We first recall the definition of torsion in the classical case: let X be a manifold and ∇ a connection on the tangent bundle T_X . The connection ∇ induces an operator of degree one on $T_X \otimes \Omega_X$, also denoted by ∇ , defined by $\nabla(\xi \otimes \omega) = \nabla(\xi) \otimes \omega + \xi \otimes d\omega$ for $\xi \in \Gamma(U, T_X)$, $\omega \in \Gamma(U, \Omega_X^n)$ for any n and any open subset $U \subset X$. Restricting to degree-one forms, one notices that $T_X \otimes \Omega_X^1 \cong T_X \otimes T_X^* \cong \text{End}(T_X)$. So, one can consider the element $\iota \in \Gamma(X, T_X \otimes \Omega_X^1)$ corresponding to the element of $\text{End}(T_X)$ which is the identity on fibers. Then, the torsion $\tau(\nabla)$ of the connection ∇ is given by $\tau(\nabla) := \nabla(\iota) \in \Gamma(X, T_X \otimes \Omega_X^2)$.

Equivalently, the torsion may be defined by $\tau(\nabla)(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - \{\xi, \eta\}$, where $\{-, -\}$ is the Lie bracket of vector fields. A connection is **torsion-free** if its torsion is zero.

We wish to imitate this in our setting. Consider the compositions

$$\begin{aligned} \mu \circ (\nabla_\ell \otimes_A \text{Id}) : M \otimes_A \Omega^n A &\xrightarrow{\nabla_\ell \otimes_A \text{Id}} \Omega^1 A \otimes_A M \otimes_A \Omega^n A \\ &\xrightarrow{\mu} M \otimes_{A^e} (\Omega^n A \otimes_A \Omega^1 A) \cong M \otimes_{A^e} \Omega^{n+1} A, \end{aligned} \quad (4.3.1)$$

$$\mu \circ \nabla_r : M \otimes_A \Omega^n A \xrightarrow{\nabla_r} M \otimes_A \Omega^{n+1} A \xrightarrow{\mu} M \otimes_{A^e} \Omega^{n+1} A. \quad (4.3.2)$$

Lemma 4.3.3. *For any connection $\nabla = (\nabla_\ell, \nabla_r)$ on an A -bimodule M , the operator*

$$(-1)^n \mu \circ (\nabla_\ell \otimes_A \text{Id}) + \mu \circ \nabla_r : M \otimes_A \Omega^n A \rightarrow M \otimes_{A^e} \Omega^{n+1} A$$

factors through the multiplication $M \otimes_A \Omega^n A \xrightarrow{\mu} M \otimes_{A^e} \Omega^n A$, yielding a well defined map $\nabla : M \otimes_{A^e} \Omega^n A \rightarrow M \otimes_{A^e} \Omega^{n+1} A$.

Proof. First, note that since $\nabla_\ell : M \rightarrow \Omega^1 A \otimes_A M$ is a right A -module map, the map $\nabla_\ell \otimes_A \text{Id} : M \otimes_A \Omega^n A \rightarrow \Omega^1 \otimes_A M \otimes_A \Omega^n A$ is well defined. Then, it remains to check that $\nabla' := (-1)^n \mu \circ (\nabla_\ell \otimes_A \text{Id}) + \mu \circ \nabla_r$ satisfies $\nabla'(am \otimes_A \omega) = \nabla'(m \otimes_A \omega a)$. This follows because $\nabla_\ell(am) - a\nabla_\ell(m) = da \otimes_A m$ and $\nabla_r(m \otimes_A \omega a) - \nabla_r(m \otimes_A \omega)a = (-1)^{|\omega|} m \otimes_A \omega da$. \square

Using the identification $(\Omega^1 A \otimes_A \mathbb{D}\text{er}(A)) \otimes_{A^e} \Omega^1 A \cong \mathbb{D}\text{er}(A) \otimes_{A^e} \Omega^2 A$, we obtain

Corollary 4.3.4. *Given any bimodule connection $\nabla = (\nabla_\ell, \nabla_r)$ on $\mathbb{D}\text{er}(A)$, there is a well defined map*

$$\begin{aligned} \mathbb{D}\text{er}(A) \otimes_{A^e} \Omega^1 A &\rightarrow \mathbb{D}\text{er}(A) \otimes_{A^e} \Omega^2 A, \\ \xi \otimes_{A^e} \omega &\mapsto -\nabla_\ell(\xi) \otimes_{A^e} \omega + \nabla_r(\xi) \otimes_{A^e} \omega + \xi \otimes_{A^e} d\omega. \end{aligned} \quad (4.3.5)$$

For the rest of the paper, we will assume that $\Omega^1 A$ is a finitely-generated projective A -bimodule.

Definition 4.3.6. Let $\iota \in \mathbb{D}\text{er}(A) \otimes_{A^e} \Omega^1 A \cong \text{End}_{A^e}(\mathbb{D}\text{er}(A))$ correspond to the identity element. If ∇ is a connection on $\mathbb{D}\text{er}(A)$, then, the torsion $\tau(\nabla) \in \mathbb{D}\text{er}(A) \otimes_{A^e} \Omega^2 A$ of ∇ is defined by $\tau(\nabla) := \nabla(\iota)$. A connection is **torsion-free** if the torsion is zero.

In this case, one can identify the space $\mathbb{D}\text{er}(A) \otimes_{A^e} \Omega^2 A$ in which the torsion is defined with two other spaces:

- (i) $\text{Hom}_{A^e \otimes_{A^e}}((\mathbb{D}\text{er}(A) \otimes \mathbb{D}\text{er}(A))_{1,2}, (\mathbb{D}\text{er}(A) \otimes A)_{\text{inn},\text{out}})$.
- (ii) $\text{Hom}_{A^e}(\Omega^1 A, \Omega^2 A)$;

In space (i), the notation 1, 2 means that the first A^e acts on the first $\mathbb{D}\text{er}(A)$ component and the second A^e acts on the second such component, while in $(\mathbb{D}\text{er}(A) \otimes A)_{\text{inn},\text{out}}$ the first A^e has the inner A^e -action and the second has the outer A^e -action.

In the following subsections, we will provide interpretations of torsion using each of these spaces. The first will provide an analogue of the classical formula $\tau(\nabla)(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - \{\xi, \eta\}$, replacing the Lie bracket with Van den Bergh's double Schouten-Nijenhuis bracket [VdB08]. The second will show equivalence with the definition of torsion for connections on Ω^1 given in [CQ95], and explain how to pass from connections on a module M to connections on its dual M^\vee .

4.4. The double and wheeled Schouten-Nijenhuis bracket. We will need the odd version of the double and wheeled Poisson bracket from Section 3.5. These are straightforward generalizations, where we now consider $|\mathbb{D}\text{er}(A)| = 1, |A| = 0$, and use *superbraidings* τ_σ (or equivalently, we use the bigrading explained in §4.1). The \mathbb{S} -module structure of $T_{\mathbf{k}}(T_A \mathbb{D}\text{er}(A))$ is defined using the same signed permutations, and hence, $\mathcal{F}(T_A \mathbb{D}\text{er}(A))$ will be defined as in (3.2.1), except using *supercommutators* rather than commutators. This will be the convention for the remainder of this section, as well as for Section 5.

The super versions will be called the double and wheeled *Schouten-Nijenhuis* brackets (the double S-N bracket was first defined in [VdB08]). We omit the details of this definition, since everything is essentially the same as before.

4.5. The formula $\tau(\nabla)(\xi, \eta) = \nabla_\xi \eta - \nabla_\eta \xi - \{\!\!\{\xi, \eta\}\!\!\}$. Let us use the notation

$$\nabla := \nabla_\ell + \nabla_r. \quad (4.5.1)$$

The goal of this section is to prove

Proposition 4.5.2.

$$\tau(\nabla)(\xi, \eta) = (\nabla_r)_\xi(\eta) - \tau_{(21)}(\nabla_\ell)_\eta(\xi) - \{\!\!\{\xi, \eta\}\!\!\}_L; \quad (4.5.3)$$

$$\tau(\nabla)(\xi, \eta) - \tau_{(21)}\tau(\nabla)(\eta, \xi) = \nabla_\xi(\eta) - \tau_{(21)}\nabla_\eta(\xi) - \{\!\!\{\xi, \eta\}\!\!\}. \quad (4.5.4)$$

The notation $(\nabla_\ell)_\xi, (\nabla_r)_\xi, \nabla_\xi$ is explained below. Note that (4.5.4) includes both (4.5.3) and the result of swapping ξ, η in (4.5.3), in the distinct components $\text{Der}(A) \otimes A$ and $A \otimes \text{Der}(A)$.

Notation 4.5.5. Define the pairing $\lrcorner : \text{Der}(A) \otimes \Omega^1 A \rightarrow A^e$ by the composition (cf. (1.7.3))

$$\text{Der}(A) \otimes \Omega^1 A \xrightarrow[\sim]{(d^*)^{-1} \otimes \text{Id}} (\Omega^1 A)^\vee \otimes \Omega^1 A \xrightarrow{\phi \otimes \omega \mapsto \phi(\omega)} A \otimes A,$$

and similarly $\Omega^1 A \otimes \text{Der}(A) \rightarrow A^e$ by identifying first $\text{Der}(A)$ with $(\Omega^1 A)^\vee$ using d^* . That is, one has $\xi \lrcorner \omega = ((d^*)^{-1} \xi)(\omega)$.

Notation 4.5.6. We denote

$$(\nabla_\ell)_\xi \eta := (\xi \otimes 1) \lrcorner \nabla_\ell(\eta); \quad (\nabla_r)_\xi \eta := (1 \otimes \xi) \lrcorner \nabla_r(\eta); \quad \nabla_\xi := (\nabla_\ell)_\xi + (\nabla_r)_\xi. \quad (4.5.7)$$

One may use the pairing \lrcorner to fix an isomorphism

$$\Omega^1 A \xrightarrow{\sim} (\text{Der}(A))^\vee : \omega \mapsto (\xi \mapsto \tau_{(21)}(\xi \lrcorner \omega)). \quad (4.5.8)$$

The $\tau_{(21)}$ is needed because $(\text{Der}(A))^\vee = \text{Hom}_{A\text{-bimod}}(\text{Der}(A), (A \otimes A)_{\text{out}})$ carries the A -bimodule structure on $\text{Der}(A)$ to the **outer** bimodule structure on $A \otimes A$. We always take A to have degree 0 with respect to the super structure, so that $\tau_{(21)}$ is the unsigned flip. This will henceforth be the assumed isomorphism $\Omega^1 A \xrightarrow{\sim} (\text{Der}(A))^\vee$. See the following Caution.

CAUTION 4.5.9. Note that the tautological contraction $\text{Der}(A)^\vee \otimes_{A^e} \text{Der}(A) \rightarrow A \otimes A$ differs from the contraction $\Omega^1 A \otimes_{A^e} (\Omega^1 A)^\vee$ via the flip: One has the commutative diagram

$$\begin{array}{ccc} \text{Der}(A)^\vee \otimes_{A^e} \text{Der}(A) & \longrightarrow & A \otimes A \\ \downarrow \sim & & \uparrow \tau_{(21)} \\ \Omega^1 A \otimes (\Omega^1 A)^\vee & \longrightarrow & A \otimes A \end{array} \quad (4.5.10)$$

By Notation (4.5.5), the contraction $\Omega^1 A \otimes_{A^e} \mathbb{D}er(A) \rightarrow A \otimes A$ uses the bottom arrow (identifying $\mathbb{D}er(A) \cong (\Omega^1 A)^\vee$); when we actually mention $(\mathbb{D}er(A))^\vee$, then one will use the top arrow.

If we have multiple components, we will use notation of the sort $(\xi_1 \otimes \xi_2) \lrcorner (\omega_1 \otimes \omega_2) = (\xi_1 \lrcorner \omega_1) \otimes (\xi_2 \lrcorner \omega_2)$ (we can replace the \otimes between ω_1 and ω_2 by \otimes_A ; then the whole element can be replaced by an element of $\Omega^2 A$, by Definition 4.2.5).

Let's make our choice of the pairing \lrcorner more explicit.

Notation 4.5.11. For any element $f \in M \otimes N$, we use the Sweedler summation notation $f = f' \otimes f''$, shorthand for $f = \sum_i f'_i \otimes f''_i$. So, for example, $(g \lrcorner f'') \otimes f'$ means $\sum_i (g \lrcorner f''_i) \otimes f'_i$. Similarly, $f \in M_1 \otimes \cdots \otimes M_n$ can be described as $f = f^{(1)} \otimes \cdots \otimes f^{(n)}$, with superscripts of (i) equivalent to i primes.

Lemma 4.5.12. *By the choice in Notation 4.5.5, one has*

$$(a\xi b) \lrcorner (c\omega d) = c(\xi \lrcorner \omega)' b \otimes a(\xi \lrcorner \omega)'' d; \quad \xi \in \mathbb{D}er(A), \omega \in \Omega^1 A. \quad (4.5.13)$$

The proof is immediate.

Proof of Proposition 4.5.2. For convenience, we use the notation $(1 \otimes x) \lrcorner (a \otimes b) := a \otimes (x \lrcorner b)$ and similarly with 1 appearing in other components, e.g., $(x \otimes 1) \lrcorner (a \otimes b) := (x \lrcorner a) \otimes b$.

We can pair $\tau(\nabla) \in \mathbb{D}er(A) \otimes_{A^e} \Omega^2 A$ with elements $\xi, \eta \in \mathbb{D}er(A)$ using the isomorphism $\Omega^2 A \cong \Omega^1 A \otimes_A \Omega^1 A$. Precisely, define $\tau(\nabla)(\xi, \eta)$ in $\mathbb{D}er(A) \otimes A$ as the image under the composition

$$\begin{aligned} \tau(\nabla) \in \mathbb{D}er(A) \otimes_{A^e} \Omega^2 A &\cong \mathbb{D}er(A) \otimes_{A^e} (\Omega^1 A \otimes_A \Omega^1 A) \\ &\xrightarrow{(1 \otimes \xi \otimes \eta) \lrcorner} \mathbb{D}er(A) \otimes_{A^e} ((A \otimes A) \otimes_A (A \otimes A)) \cong \mathbb{D}er(A) \otimes A. \end{aligned} \quad (4.5.14)$$

Recall that $\iota \in \mathbb{D}er(A) \otimes_{A^e} \Omega^1 A$ corresponds to $\text{Id} \in \text{End}_{A^e}(\mathbb{D}er(A))$ (and $\text{Id} \in \text{End}_{A^e}(\Omega^1 A)$). Let us suppose that $\iota = \sum_s \xi_s \otimes_{A^e} \omega_s$ for some $\xi_s \in \mathbb{D}er(A)$ and $\omega_s \in \Omega^1 A$. We will need to use the resulting identities

$$\sum_s (\xi_s \lrcorner \omega_s)'' \cdot \xi_s \cdot (\xi_s \lrcorner \omega_s)' = \xi, \quad \sum_s (\xi_s \lrcorner \omega_s)' \cdot \omega_s \cdot (\xi_s \lrcorner \omega_s)'' = \omega, \quad (4.5.15)$$

which follow from definitions (see (4.5.13)). For any element $\xi \in \mathbb{D}er(A)$, let us use the notation $\theta_\xi : A \rightarrow A \otimes A$ for the associated map (to avoid confusion with multiplying by ξ in $T_A \mathbb{D}er(A)$). We then have, using the natural identifications $(\mathbb{D}er(A)) \otimes_A (A \otimes A) \cong \mathbb{D}er(A) \otimes A$, and $(A \otimes A) \otimes_A \mathbb{D}er(A) \cong A \otimes \mathbb{D}er(A)$:

$$\begin{aligned} \tau(\nabla)(\xi, \eta) &= - \sum_s \tau_{21} \circ ((\eta \otimes 1) \lrcorner \nabla_\ell(\xi_s)) \otimes_{A^e} (\xi_s \lrcorner \omega_s) \\ &\quad + ((1 \otimes \xi) \lrcorner \nabla_r(\xi_s)) \otimes_{A^e} (\eta \lrcorner \omega_s) + \xi_s \otimes_{A^e} ((\xi \otimes \eta) \lrcorner d\omega_s), \end{aligned} \quad (4.5.16)$$

$$- \sum_s ((\eta \otimes 1) \lrcorner \nabla_\ell(\xi_s)) \otimes_{A^e} (\xi_s \lrcorner \omega_s) = -(\nabla_\ell)_\eta \xi + \sum_s \theta_\eta((\xi_s \lrcorner \omega_s)'') \otimes_A \xi_s \cdot (\xi_s \lrcorner \omega_s)', \quad (4.5.17)$$

and

$$\sum_s ((1 \otimes \xi) \lrcorner \nabla_r(\xi_s)) \otimes_{A^e} (\eta \lrcorner \omega_s) = (\nabla_r)_\xi \eta - \sum_s (\eta \lrcorner \omega_s)'' \xi_s \otimes_A \theta_\xi((\eta \lrcorner \omega_s)'). \quad (4.5.18)$$

Let $i_1 : (A \otimes A)_{\text{inn}} \otimes_{A^e} (A \otimes A \otimes A) \xrightarrow{\sim} A \otimes A \otimes A$ be given by $i_1((a \otimes b)_{\text{inn}} \otimes_{A^e} (c \otimes e \otimes f)) = ac \otimes fb \otimes e$, which is what is needed to have the commutative diagram

$$\begin{array}{ccccc} \mathbb{D}\text{er}(A) \otimes_{A^e} (A \otimes A \otimes A) & \xrightarrow{\sim} & \mathbb{D}\text{er}(A) \otimes A & \xrightarrow{\lrcorner (\omega \otimes 1)} & A \otimes A \otimes A \\ & \searrow \lrcorner (\omega \otimes 1) & & \nearrow i_1 & \\ & & (A \otimes A)_{\text{inn}} \otimes_{A^e} (A \otimes A \otimes A) & & \end{array} \quad (4.5.19)$$

Then one has:

$$\begin{aligned} & \sum_s i_1((\xi_s \lrcorner \omega)_{\text{inn}} \otimes_{A^e} ((\xi \otimes \eta) \lrcorner d\omega_s)) = \tau_{(32)}((\xi \otimes \eta) \lrcorner d\omega \\ & - \sum_s (\theta_\xi((\xi_s \lrcorner \omega)') \otimes_A \eta \lrcorner \omega_s (\xi_s \lrcorner \omega)'' + (\xi_s \lrcorner \omega)' \xi \lrcorner \omega_s \otimes_A \theta_\eta((\xi_s \lrcorner \omega)'))). \end{aligned} \quad (4.5.20)$$

Contracting (4.5.17), (4.5.18) with ω and adding (4.5.20), one obtains

$$\begin{aligned} & (-\tau(\nabla)(\xi, \eta) + (\nabla_r)_\xi(\eta) - \tau_{(21)}(\nabla_\ell)_\eta(\xi)) \lrcorner \omega \\ & = \sum_s \tau_{(32)} \left(-(\xi \otimes \eta) \lrcorner d\omega + (\theta_\xi((\xi_s \lrcorner \omega)'(\eta \lrcorner \omega_s)') \otimes (\eta \lrcorner \omega_s)''(\xi_s \lrcorner \omega)'' \right. \\ & \quad \left. - ((\xi_s \lrcorner \omega)'(\xi \lrcorner \omega_s)' \otimes \theta_\eta((\xi \lrcorner \omega_s)''(\xi_s \lrcorner \omega)')) \right) \\ & = \tau_{(32)}(-(\xi \otimes \eta) \lrcorner d\omega + (\theta_\xi \otimes 1)(\eta \lrcorner \omega) - (1 \otimes \theta_\eta)(\xi \lrcorner \omega)) = \llbracket \xi, \eta \rrbracket_L \lrcorner (\omega \otimes 1). \end{aligned} \quad (4.5.21)$$

For the last equality, we note that $\llbracket \xi, \eta \rrbracket_L$ is defined by $\llbracket \xi, \eta \rrbracket_L \lrcorner (da \otimes 1) = \tau_{(32)}((\theta_\xi \otimes 1) \circ \theta_\eta(a) - (1 \otimes \theta_\eta) \circ \theta_\xi(a))$; there is then a unique extension of $\llbracket \xi, \eta \rrbracket_L$ to a map $\Omega^1 A \otimes A \rightarrow A \otimes A \otimes A$ as indicated in the last line. \square

4.6. Torsion of connections on Ω^1 . Note: This subsection and the next will not be used elsewhere in this paper.

It turns out that torsion of connections on Ω^1 is easier to define. In the classical setting, a connection ∇ on Ω_X^1 for a manifold X is a map $\nabla : \Omega_X^1 \rightarrow \Omega_X^1 \otimes \Omega_X^1$ which is a derivation in the sense that $\nabla(a\omega) = a\nabla(\omega) + \omega \otimes da$, for local sections ω and a of Ω_X^1 and \mathcal{O}_X , respectively. Then, to compare with $d : \Omega_X^1 \rightarrow \Omega_X^2 \cong \Lambda^2 \Omega_X^1$, let us define $q : \Omega_X^1 \otimes \Omega_X^1 \rightarrow \Omega_X^2$ to be the quotient. Then we may consider

$$\tau(\nabla) := q \circ \nabla + d : \Omega_X^1 \rightarrow \Omega_X^2, \quad (4.6.1)$$

which we can call the **torsion** of ∇ .

The definition of torsion in [CQ95] for connections on $\Omega^1 A$ is then the noncommutative analogue of the above. Namely, in the noncommutative case $\Omega^2 A = \Omega^1 A \otimes_A \Omega^1 A$, so ∇_r, ∇_ℓ , and d are all maps $\Omega^1 \rightarrow \Omega^2 A$. To get an A^e -module map, it is clear that one considers the combination

$$\tau(\nabla) := -\nabla_\ell + \nabla_r + d, \quad (4.6.2)$$

which is called the **torsion** of ∇ .

Remark 4.6.3. As is mentioned in [CQ95], given any choice of torsion and left or right connection, (4.6.2) defines a right or left connection: so there are one-to-one correspondences

$$\text{left connections on } \Omega^1 A \xleftrightarrow{\nabla_\ell \mapsto \nabla_\ell - d} \text{right connections on } \Omega^1 A \quad (4.6.4)$$

and

$$\text{left connections on } \Omega^1 A \text{ with choice of torsion} \longleftrightarrow \text{connections on } \Omega^1 A. \quad (4.6.5)$$

One may easily see the corresponding statement for connections on $\mathbb{D}er(A)$.

4.7. Dual connections and equivalence of torsion with dual torsion. Classically, given a connection $\nabla : E \rightarrow E \otimes \Omega_X^1$ on a vector bundle E , one may define a natural dual connection ∇^\vee on E^\vee , such that, if E is of finite-rank, $(\nabla^\vee)^\vee \cong \nabla$ under the canonical isomorphism $E \cong (E^\vee)^\vee$. Namely, one uses the formula (for $f \in \Gamma(U, E^\vee), e \in \Gamma(U, E)$):

$$\nabla^\vee(f) \lrcorner e := d(f \lrcorner e) - (f \otimes 1) \lrcorner \nabla(e). \quad (4.7.1)$$

For a (projective) bimodule M over a ring A with connection $\nabla = (\nabla_\ell, \nabla_r)$, one may define a dual connection ∇^\vee on $M^\vee := \text{Hom}_{A^e}(M, A^e)$. Let us define the pairing $\lrcorner : M^\vee \otimes M \rightarrow A$ by applying M^\vee to M , which means that one has (4.5.13), considering $\xi \in M^\vee$ and $\omega \in M$.

Let us first try to dualize ∇_ℓ . It is natural to consider the two possible compositions $M^\vee \otimes M \xrightarrow{\lrcorner} A \otimes A \xrightarrow{d \otimes 1} \Omega^1 A \otimes A$ and $M^\vee \otimes M \xrightarrow{\lrcorner} A \otimes A \xrightarrow{1 \otimes d} A \otimes \Omega^1 A$. As for using the connection ∇_ℓ , it is natural to consider $M^\vee \otimes M \xrightarrow{1 \otimes \nabla_\ell} M^\vee \otimes \Omega^1 A \otimes_A M \xrightarrow{\lrcorner} (\Omega^1 A \otimes_A A) \otimes A \cong \Omega^1 A \otimes A$.

In the end, we will need something that is A^e -linear in M (so as to get a map $M^\vee \rightarrow M^\vee \otimes \Omega^1 A$ or $M^\vee \rightarrow \Omega^1 A \otimes M^\vee$). As the latter map is right A -linear in M and a derivation on the left, we need to consider

$$((d \otimes 1) \circ \lrcorner) - (\lrcorner \circ (1 \otimes \nabla_\ell)) : M^\vee \otimes M \rightarrow \Omega^1 A \otimes A. \quad (4.7.2)$$

In M^\vee , this is left A -linear and a derivation on the right. Hence the resulting map should be considered as a **right** connection: dualizing left connections results in right connections. Finally, we define the operations $\lrcorner m : \Omega^1 A \otimes_A M^\vee \rightarrow A \otimes \Omega^1 A, M^\vee \otimes_A \Omega^1 A \rightarrow \Omega^1 A \otimes A$ by $(\omega \otimes_A f) \lrcorner m = (f \lrcorner m)' \otimes \omega(f \lrcorner m)''$, $(f \otimes_A \omega) \lrcorner m = (f \lrcorner m)' \omega \otimes (f \lrcorner m)''$. We then have the

Lemma-Definition 4.7.3. *For any left connection $\nabla_\ell : M \rightarrow \Omega^1 A \otimes_A M$ on an A^e -module M , the map $\nabla_r^\vee : M^\vee \rightarrow M^\vee \otimes_A \Omega^1 A$ given by*

$$\nabla_r^\vee(f) \lrcorner m = ((d \otimes 1)(f \lrcorner m) - \tau_{(21)}(1 \otimes f) \lrcorner \nabla_\ell(m)), \quad (\text{given } \nabla_\ell, \text{ a left conn.}) \quad (4.7.4)$$

is a right connection. Similarly, if ∇_r is right connection,

$$\nabla_\ell^\vee(f) \lrcorner m = ((1 \otimes d)(f \lrcorner m) - \tau_{(21)}(f \otimes 1) \lrcorner \nabla_r(m)), \quad (\text{given } \nabla_r, \text{ a right conn.}) \quad (4.7.5)$$

defines a left connection. So, for any bimodule connection $\nabla = (\nabla_\ell, \nabla_r)$, one may define a dual connection $\nabla^\vee = (\nabla_\ell^\vee, \nabla_r^\vee)$ by (4.7.4), (4.7.5).

Note that, if $m \mapsto m^{\vee\vee}$ under $M \xrightarrow{\sim} M^{\vee\vee}$, and $f \in M^\vee$, then $f \lrcorner m = \tau_{(21)}(m^{\vee\vee} \lrcorner f)$. It is then immediate that

Lemma 4.7.6. *If M is finitely-generated (projective), then $(\nabla^\vee)^\vee \cong \nabla$ under the natural isomorphism $(M^\vee)^\vee \cong M$.*

We now prove that, in both the classical and the noncommutative cases, torsion of a connection on T_X ($\mathbb{D}\text{er}(A)$) and of a connection on Ω_X^1 ($\Omega^1 A$) are identical (under the appropriate natural identifications of spaces).

Proposition 4.7.7. *Let ∇ be a connection on Ω_X^1 and ∇^\vee its dual connection on T_X . Then $\tau(\nabla) = q \circ \nabla + d : \Omega_X^1 \rightarrow \Omega_X^2$ is naturally identified with $\tau(\nabla^\vee) = (\nabla^\vee \otimes 1)(\iota)$ in $\Gamma(X, T_X \otimes \Omega_X^2)$ (or with $\tau(\nabla^\vee) : T_X^2 \rightarrow T_X$ given by $\tau(\nabla^\vee)(\xi, \eta) = \nabla_\xi^\vee \eta - \nabla_\eta^\vee \xi - \{\xi, \eta\}$).*

Proof. We show equivalence of $q \circ \nabla + d$ with $\tau(\nabla^\vee)(\xi, \eta) = \nabla_\xi^\vee \eta - \nabla_\eta^\vee \xi - \{\xi, \eta\}$. Let us consider

$$(q \circ \nabla(\omega) + d\omega)(\xi, \eta) = \xi \lrcorner (\nabla_\eta \omega) - \eta \lrcorner (\nabla_\xi \omega) + \xi(\eta \lrcorner \omega) - \eta(\xi \lrcorner \omega) - \{\xi, \eta\} \lrcorner \omega, \quad (4.7.8)$$

while

$$(\nabla_\xi^\vee \eta - \nabla_\eta^\vee \xi) \lrcorner (\omega \otimes 1) = \xi(\eta \lrcorner \omega) - \eta(\xi \lrcorner \omega) - \eta \lrcorner \nabla_\xi \omega + \xi \lrcorner \nabla_\eta \omega, \quad (4.7.9)$$

proving the desired result. \square

Proposition 4.7.10. *Let ∇ be a bimodule connection on $\Omega^1 A$ and ∇^\vee the dual connection on $\mathbb{D}\text{er}(A)$. Then $\tau(\nabla) : \Omega^1 A \rightarrow \Omega^2 A$ is naturally identified with $\tau(\nabla^\vee) \in \mathbb{D}\text{er}(A) \otimes_{A^e} \Omega^2 A \cong \text{Hom}_{A^e \otimes A^e}((\mathbb{D}\text{er}(A) \otimes \mathbb{D}\text{er}(A))_{1,2}, (\mathbb{D}\text{er}(A) \otimes A)_{\text{inn,out}})$, where the last space is where the formula (4.5.3) lives (see the end of Section 4.3).*

Proof. As before, we show equivalence of $-\nabla_\ell + \nabla_r + d$ with $\xi \otimes \eta \mapsto (\nabla_r^\vee)_\xi \eta - \tau_{(21)}(\nabla_\ell^\vee)_\eta \xi - \{\xi, \eta\}_\ell$. We first note that $f \in \text{Hom}_{A^e}(\Omega^1 A, \Omega^2 A)$ corresponds to $f' \in \text{Hom}_{A^e \otimes A^e}((\mathbb{D}\text{er}(A) \otimes \mathbb{D}\text{er}(A))_{1,2}, (\mathbb{D}\text{er}(A) \otimes A)_{\text{inn,out}})$ if and only if we have $(\xi \otimes \eta) \lrcorner (f(\omega)) = \tau_{(32)} f'(\xi \otimes \eta) \lrcorner (\omega \otimes 1)$, which can be checked (for example) by looking at how each side of the formula changes when ξ, η , and ω are acted on by A^e .

Now, we compute

$$(\xi \otimes \eta) \lrcorner (-\nabla_\ell(\omega) + \nabla_r(\omega) + d\omega) = -(1 \otimes \eta) \lrcorner (\nabla_\ell)_\xi(\omega) + (\xi \otimes 1) \lrcorner (\nabla_r)_\eta(\omega) \\ + \theta_{\xi \otimes 1}(\eta \lrcorner \omega) - \theta_{1 \otimes \eta}(\xi \lrcorner \omega) - \tau_{(32)}(\{\xi, \eta\}_\ell \lrcorner (\omega \otimes 1)), \quad (4.7.11)$$

where we used the last line of (4.5.21) to expand $(\xi \otimes \eta) \lrcorner d\omega$.

Now, let us expand $\tau(\nabla^\vee)(\xi, \eta) + \{\xi, \eta\}_\ell = (\nabla_r^\vee)_\xi \eta - \tau_{(21)}(\nabla_\ell^\vee)_\eta \xi$ applied to $\omega \otimes 1$:

$$((\nabla_r^\vee)_\xi \eta - \tau_{(21)}(\nabla_\ell^\vee)_\eta \xi) \lrcorner (\omega \otimes 1) = \tau_{(32)}(\xi \lrcorner (\nabla_r^\vee)_\eta \omega - \eta \lrcorner (\nabla_\ell^\vee)_\xi \omega) \\ = \tau_{(32)}(\theta_{\xi \otimes 1}(\eta \lrcorner \omega) - (1 \otimes \eta) \lrcorner (\nabla_\ell)_\xi(\omega) - \theta_{1 \otimes \eta}(\xi \lrcorner \omega) + (\xi \otimes 1) \lrcorner (\nabla_r)_\eta(\omega)), \quad (4.7.12)$$

which proves that $\tau_{(32)}(\tau(\nabla^\vee)(\xi, \eta) \lrcorner (\omega \otimes 1)) =$ the RHS of (4.7.11), as desired. \square

5. The BV operator D_∇

5.1. The classical story. In this subsection we briefly recall a classical construction, following [Kos85], of BV structures on ΛT_X for a finite-dimensional smooth manifold X , which generate the Schouten-Nijenhuis bracket.

Let ∇ be a connection on T_X . The connection extends to a connection $\nabla : \Lambda^n T_X \rightarrow \Lambda^n T_X \otimes \Omega_X^1$ satisfying $\nabla(\xi \wedge \eta) = \nabla(\xi) \wedge \eta + \xi \wedge \nabla(\eta)$. (Note that the derivation property of ∇ does not allow one to put a sign such as $(-1)^{|\xi|}$ in front of the second term.)

Let $\iota \in \Gamma(X, T_X \otimes \Omega_X^1)$ be the canonical section which is the identity on fibers. Given a section of $\Lambda^n T_X \otimes \Lambda^m \Omega_X^1$, one may consider the contraction i_ι with ι : this means

$$i_\iota(\xi_1 \wedge \cdots \wedge \xi_n \wedge \omega_1 \wedge \cdots \wedge \omega_m) \\ = \sum_{j=1}^n \sum_{\ell=1}^m (-1)^{i+j} i_\iota(\xi_i \otimes \omega_j) \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_n \wedge \omega_1 \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_m. \quad (5.1.1)$$

One may then consider

$$D_\nabla := i_\iota \circ \nabla : \Lambda^n T_X \rightarrow \Lambda^{n-1} T_X, \quad (5.1.2)$$

which can also be written as

$$D_\nabla = \sum_s i_{\omega_s} \nabla_{\xi_s}, \quad \iota = \sum_s \xi_s \otimes \omega_s. \quad (5.1.3)$$

It is easy to see that D_∇ is a differential operator of order ≤ 2 . Since ∇ is torsion-free if and only if $\nabla_\xi \eta - \nabla_\eta \xi = \{\xi, \eta\}$ for all $\xi, \eta \in \Gamma(U, T_X)$ (where $U \subset X$ denotes any open subset), one may easily show

Proposition 5.1.4. [Kos85] *The connection ∇ is torsion-free if and only if the principal symbol of D_∇ (as an operator of order 2) is \pm the Schouten-Nijenhuis bracket; precisely,*

$$D_\nabla(\xi \wedge \eta) - \xi D_\nabla(\eta) - D_\nabla(\xi)\eta = (-1)^{|\xi|+1}\{\xi, \eta\} \text{ for all } \xi, \eta \in \Gamma(U, \Lambda T_X). \quad (5.1.5)$$

Furthermore, one may get a formula for D_∇^2 . First, [Kos85] remarks that $D_\nabla^2 = \frac{1}{2}[D_\nabla, D_\nabla]$ where $[\cdot, \cdot]$ is the supercommutator (using degree of differential operator, *not* order: D_∇ lowers degree by one), which shows that D_∇^2 has order $\leq 3 = 2 + 2 - 1$ as a differential operator (because D_∇ itself has order ≤ 2). Then, [Kos85] computes the principal symbol $\Gamma_3(D_\nabla^2)$ as an operator of degree ≤ 3 , in terms of the Jacobiator of the principal symbol $\Gamma_2(D_\nabla)$ (as an operator of degree ≤ 2). Namely, $\Gamma_3(D_\nabla^2) = \Gamma_3(\frac{1}{2}[D_\nabla, D_\nabla]) = \frac{1}{2}[\Gamma_2[D_\nabla], \Gamma_2[D_\nabla]]$. This means that, if $\Gamma_2(D_\nabla)$ satisfies the Jacobi identity, then $\Gamma_3(D_\nabla^2) = 0$, so D_∇^2 is actually an operator of degree ≤ 2 .

Now, ∇ is torsion-free if and only if $\Gamma_2(D_\nabla)$ is the Schouten-Nijenhuis bracket. In this case, D_∇^2 must have order ≤ 2 . Since it is also an operator of degree -2 , it must be contraction with a two-form.

It remains to compute this two-form. Koszul gives the following formula:

Proposition 5.1.6. [Kos85] *One has*

$$D_\nabla^2 = i_{\text{tr}(\nabla^2)}, \quad (5.1.7)$$

where $\nabla^2 : T_X \rightarrow T_X \otimes \Omega_X^2$ is the curvature and $\text{tr}(\nabla^2) \in \Gamma(X, \Omega_X^2)$ is its trace, which can also be written as $\text{tr}(\nabla^2) : \Lambda^{\dim X} T_X \xrightarrow{\nabla^2} \Lambda^{\dim X} T_X \otimes \Omega_X^2$.

In particular, D_∇ gives a BV structure generating the Schouten-Nijenhuis bracket in the case that $\text{tr}(\nabla^2) = 0$ and ∇ is torsion-free.

In Koszul's paper, the verification of (5.1.7) is omitted; this seems to be the hardest technical part of the proof. In §5.8, we will give a new proof of (5.1.7), since we will need to apply the same proof to the noncommutative setting. Our proof actually works in the smooth algebraic setting, and is based on purely global arguments on any affine variety, replacing the tangent bundle by its global sections $(\text{Der}(A), \text{ where } X = \text{Spec } A)$, viewed as a projective module.

Let us return to Koszul's setting. Note that the D_∇ are all the BV structures possible which generate the Schouten-Nijenhuis bracket: any two differential operators of order 2 with the same principal symbol must differ by an operator of order ≤ 1 (a derivation). If, furthermore, the operators have degree -1 (as is the case here), then the resulting difference must be given by an \mathcal{O}_X -linear map $T_X \rightarrow \mathcal{O}_X$, i.e., contraction with a global one-form ω . Similarly, two torsion-free connections differ by a linear map $T_X \rightarrow T_X \otimes \Omega_X^1$, i.e., a global $\text{End}(T_X)$ -valued one-form β , such that $(\eta \lrcorner \beta)(\xi) = (\xi \lrcorner \beta)(\eta)$ for all $\xi, \eta \in \Gamma(X, T_X)$. It remains only to show that one can produce, for every global section $\omega \in \Gamma(X, \Omega_X^1)$, such a one-form β , so that $\text{tr}(\beta) = \omega$. This is clear. Thus, one can always go from a differential operator D with principal symbol $[-, -]$ to a connection ∇ such that $D = D_\nabla$.

Finally, it is evident by (5.1.5) that D_∇ is determined by its restriction to vector fields, where one sees that

$$D_\nabla|_{T_X} = \operatorname{div} \nabla := \operatorname{tr} \circ \nabla : T_X \xrightarrow{\nabla} T_X \otimes \Omega_X^1 \xrightarrow{\lrcorner} \mathcal{O}_X, \quad (5.1.8)$$

where $\lrcorner : T_X \otimes \Omega_X^1 \rightarrow \mathcal{O}_X$ is the contraction. Hence, two connections ∇, ∇' induce the same differential operator if and only if they have the same divergence: $\operatorname{div} \nabla = \operatorname{div} \nabla'$. (One may also express $\operatorname{tr} \circ \nabla$ as the trace of the \mathcal{O}_X -linear map $T_X \rightarrow \operatorname{End}_k(T_X) \cong T_X \otimes \Omega_X^1, \xi \mapsto \nabla_\xi(-)$.) Summarizing:

Proposition 5.1.9. [Kos85] *The map $\nabla \rightarrow D_\nabla$ gives a one-to-one correspondence between torsion-free connections up to equivalence, and differential operators of order 2 on T_X whose principal symbol is \pm the Schouten-Nijenhuis bracket. Here ∇, ∇' are “equivalent” if and only if they have the same divergence. Furthermore, torsion-free connections which are trace-flat ($\operatorname{tr}(\nabla^2) = 0$) correspond to BV structures generating the Schouten-Nijenhuis bracket.*

5.2. BV and Gerstenhaber wheelgebras. Now, we pursue a wheeled analogue of the preceding section. The first step is to define BV and Gerstenhaber wheelgebras: these are just BV and Gerstenhaber algebras in the category of wheelspaces.

To interpret BV wheelgebras in terms of differential operators on a Gerstenhaber wheelgebra (which is, in particular, a supercommutative wheelgebra), we need to apply the categorical definition of differential operators to the category of super wheelspaces: these are wheelspaces equipped with an additional \mathbb{Z} -grading, with the super (i.e., signed) braidings. We will use “degree” to refer to this new degree, and “wheeled degree” to refer to the wheelspace degree.

We thus obtain an almost-supercommutative wheelgebra of differential operators. Explicitly, D has degree m if $|D(x)| = |x| + m$ for any homogeneous x , and then $[D, D'] := D \circ D' - (-1)^{|D||D'|} D' \circ D$.

5.3. Overview of wheeled version of D_∇ . We would like to mimic Section 5.1 in the wheeled (noncommutative geometry) context, by replacing classical notions with the twisted versions of Section 5.2. To do this, we first need to establish some preliminaries concerning multilinearity and connections in the noncommutative case. To demonstrate how this is important, let us begin with a bimodule connection $\nabla = (\nabla_\ell, \nabla_r)$ on $\mathbb{D}\operatorname{er}(A)$. We note that ∇ induces a “connection” on $T_A \mathbb{D}\operatorname{er}(A)$ as follows (recall (4.5.1)):

$$\nabla : T_A^n \mathbb{D}\operatorname{er}(A) \rightarrow \bigoplus_{0 \leq i \leq n} T_A^i \mathbb{D}\operatorname{er}(A) \otimes_A \Omega^1 A \otimes_A T_A^{n-i} \mathbb{D}\operatorname{er}(A); \quad (5.3.1)$$

$$\nabla : \xi_1 \otimes_A \cdots \otimes_A \xi_n \mapsto \sum_i \xi_1 \otimes_A \cdots \otimes_A \nabla(\xi_i) \otimes_A \cdots \otimes_A \xi_n. \quad (5.3.2)$$

This is well defined because ∇_r in the i -th component and ∇_ℓ in the $i+1$ -th component are compatible in the sense that

$$\nabla_r(\xi a) \otimes_A \eta + (\xi a) \otimes_A \nabla_\ell(\eta) = \nabla_r(\xi) \otimes_A (a\eta) + \xi \otimes_A \nabla_\ell(a\eta). \quad (5.3.3)$$

Then, we would like to define D_∇ by $i_\ell \circ \nabla$. However, when we contract with ι , the result does not live in $T_A \mathbb{D}\mathrm{er}(A)$ anymore: one may see (by Corollary 5.6.9) that the result makes sense in a space of the form $T_A \mathbb{D}\mathrm{er}(A) \otimes (T_A \mathbb{D}\mathrm{er}(A)/[T_A \mathbb{D}\mathrm{er}(A), T_A \mathbb{D}\mathrm{er}(A)])$.

To iterate, we can proceed by using the space $\mathcal{F}(T_A \mathbb{D}\mathrm{er}(A))$. For example, the connection ∇ descends to a well defined map

$$\nabla : (T_A \mathbb{D}\mathrm{er}(A))_{\mathrm{cyc}} \rightarrow \bigoplus_{0 \leq i \leq n} T_A \mathbb{D}\mathrm{er}(A) \otimes_{A^e} \Omega^1 A. \quad (5.3.4)$$

On this latter sum, we may still define the contraction i_ℓ , which lands back in $\mathcal{F}(T_A \mathbb{D}\mathrm{er}(A))$.

5.4. The wheeled Schouten-Nijenhuis bracket. The wheeled Schouten-Nijenhuis bracket is the super version of the wheeled Poisson bracket from Definition 3.5.17, with $|\mathbb{D}\mathrm{er}(A)| = 1, |A| = 0$. (This gives the structure of a Gerstenhaber wheelgebra).

In particular, one obtains an ordinary Gerstenhaber bracket on $\mathcal{F}_0(\mathbb{D}\mathrm{er}(A))$. This identifies with the supersymmetric algebra on $(T_A \mathbb{D}\mathrm{er}(A))_{\mathrm{cyc}}$. The latter Lie algebra, noticed in [VdB08], is a generalization of the “necklace Lie algebra,” defined in [BLB02, Gin01].

5.5. Connections and bimodule contractions on spaces $\mathcal{F}(T_A M)$. In this section, we will use *bimodule contractions* to refer to operations of the sort $\xi \lrcorner \omega$, for $\xi \in \mathbb{D}\mathrm{er}(A), \omega \in \Omega^1 A$, or more generally, $\xi \in M, \omega \in M^\vee$, for any finitely-generated projective bimodule M . In contrast, *wheeled contractions* refer to the contractions $\mu_{i,j}$ that are part of the structure of wheelspaces.

Suppose we are given an A -bimodule, M , and an A -bimodule connection $\nabla = (\nabla_\ell, \nabla_r)$. We would like to say that ∇ extends to an operator $\mathcal{F}_m(T_A M) \rightarrow \mathcal{F}_m(T_A(M \oplus \Omega^1 A))$, which commutes with wheeled contractions and permutations, and lands in the subspace of degree one in $\Omega^1 A$. It suffices to define the operator ∇ on $T_A M, T_A M/[T_A M, T_A M]$ and extend by

$$\nabla(a \otimes b) = \nabla(a) \otimes b + a \otimes \nabla(b), \quad \nabla(\sigma \otimes a) = \sigma \otimes \nabla(a). \quad (5.5.1)$$

We have

$$\nabla(m_1 \otimes_A \cdots \otimes_A m_n) = \sum_{i=1}^n (m_1 \otimes_A \cdots \otimes_A (\nabla_\ell + \nabla_r)(m_i) \otimes_A \cdots \otimes_A m_n). \quad (5.5.2)$$

and we obtain the formula on $\mathcal{F}_0(T_A M)$ by taking a wheeled contraction of this.

It is not difficult to check that ∇ extends to an operator on $\mathcal{F}(T_A(M \oplus \Omega^1 A))$, which commutes with wheeled contractions and permutations, as follows. Let us use superbraidings with respect to a bigrading with $|A| = (0, 0)$, $|\Omega^1 A| = (0, -1)$, and $|M| \subset \mathbb{Z} \times \{0\}$.

Lemma-Definition 5.5.3. *The operator $\nabla : T_{\mathbf{k}}(M \oplus A \oplus \Omega^1 A) \rightarrow \mathcal{F}(M \oplus \Omega^1 A)$ given by $\nabla : f_1 \otimes \cdots \otimes f_m \rightarrow \sum_{i=1}^m (-1)^{|\{j < i : f_j \in \Omega^1 A\}|} \nabla^{(i)}(f_1 \otimes \cdots \otimes f_m)$, where*

$$\nabla(f) := \begin{cases} \nabla_\ell(f) + \nabla_r(f), & a \in M, \\ df, & f \in A \text{ or } \Omega^1 A, \end{cases} \quad (5.5.4)$$

extends uniquely to an endomorphism of wheelspaces of $\mathcal{F}(M \oplus \Omega^1 A)$. Hence, we define ∇ in this way.

Next, we need to define bimodule contractions. Suppose that M, M^\vee are dual finitely-generated projective A -bimodules (i.e., $M^\vee \xrightarrow{\sim} \text{Hom}_{A^e}(M, A^e)$). Then we have a map $i_{pre} : M^{\otimes n} \otimes M^\vee \rightarrow M^{\otimes(n-1)} \otimes (A \otimes A)$, by contracting the last two factors. This map is $(A^e)^{\otimes(n+1)}$ -linear, sending the A^e -action on the m -th component to the A^e -action on the m -th component on the right hand side for $m \leq n$, if one considers the $(A \otimes A)$ term as a single component with outer action; then, the A^e action on M^\vee gets sent to the inner A^e -action on the $A \otimes A$ term.

From this, we may consider a contraction

$$(\text{Ind}_{S_m}^{S_m \times S_m} M^{\otimes m}) \otimes (\text{Ind}_{S_n}^{S_n \times S_n} (M^\vee)^{\otimes n}) \rightarrow (\text{Ind}_{S_{m+n}}^{S_{m+n} \times S_{m+n}} (M^{\otimes(n-m)} \otimes A^{\otimes 2m})), \quad (5.5.5)$$

if $m \leq n$, by total contraction, as follows. By $\text{Ind}_{S_m}^{S_m \times S_m}$, we mean induced with respect to the diagonal embedding, and we will write the action of $S_m \times S_m$ on an element x with the notation $\sigma \cdot x \cdot \tau$ for $\sigma, \tau \in S_m$. Now, for any $\sigma_L(a_1 \otimes \cdots \otimes a_m) \sigma_R$ and $\sigma'_L(a'_1 \otimes \cdots \otimes a'_n) \sigma'_R$,

1. Tensor the two elements, giving $(\sigma_L \times \sigma'_L)(a_1 \otimes \cdots \otimes a_m \otimes a'_1 \otimes \cdots \otimes a'_n)(\sigma_R \times \sigma'_R)$.
2. Summing over all ways to rewrite the above expression by applying a cyclic permutation to σ_L and cyclically permuting the a_i , contract the two adjacent elements $M \otimes M^\vee$ that appear, sending the A^e -bimodule structure on M to the outer structure (and the A -bimodule structure on M^\vee to the inner structure) on $A \otimes A$. This yields

$$\sum_{i=1}^m (12 \cdots m)^{-i} (\sigma_L \times \sigma'_L) \otimes (a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{i-1} \otimes (a'_1(a_i) \otimes a'_2 \otimes \cdots \otimes a'_n) \otimes (m, m+1)(12 \cdots m)^{-i} (\sigma_R \times \sigma'_R). \quad (5.5.6)$$

3. Rewrite the above by moving the $(A \otimes A)$ -components all the way to the right, and adding appropriate permutations to the left and right of the above expression. Continue iterating until one has an element of $\text{Ind}_{S_{n+m}}^{S_{n+m} \times S_{n+m}} (M^{\otimes n-m} \otimes A^{\otimes 2m})$.

Proposition 5.5.7. (i) *The above procedure is well defined, and using wheeled contractions, yields **total contraction maps***

$$\begin{aligned} i_{tot} : \mathcal{F}_m(T_A M) \otimes \mathcal{F}_n(T_A M^\vee) &\rightarrow \mathcal{F}_{n+m}(T_A M) \oplus_{\mathcal{F}_{n+m}(A)} \mathcal{F}_{n+m}(T_A M^\vee) \\ &\subset \mathcal{F}_{n+m}(T_A(M \oplus M^\vee)), \end{aligned} \quad (5.5.8)$$

which respect the grading, $|M| = 1, |A| = 0, |M^\vee| = -1$, and the $(A^e)^{\otimes(m+n)}$ -module structure.

(ii) *The above map extends, for any N, N' , to a map*

$$\mathcal{F}_m(T_A(M \oplus N)) \otimes \mathcal{F}_n(T_A(M^\vee \oplus N')) \rightarrow \mathcal{F}_{m+n}(T_A(M \oplus M^\vee \oplus N \oplus N')),$$

that respects the above grading and has image contained in the sum of tensors which have only occurrences of M or M^\vee , but not both.

Given $f \in \mathcal{F}_n(T_A M^\vee)$, let $i_f : \mathcal{F}_m(T_A M) \rightarrow \mathcal{F}_{n+m}(T_A M) + \mathcal{F}_{n+m}(T_A M^\vee)$ be the map $i_{tot}(- \otimes f)$, and denote the induced map $\mathcal{F}_m(T_A(M \oplus N)) \rightarrow \mathcal{F}_{n+m}(T_A(M \oplus M^\vee \oplus N))$ by i_f as well.

Finally, we define trace. Let M, M_1, M_2 be any A -bimodules.

Definition 5.5.9. Given an A -bimodule map $\phi : M \rightarrow M_1 \otimes_A M \otimes_A M_2$, where M is projective, let $\phi' \in M^\vee \otimes_{A^e} (M_1 \otimes_A M \otimes_A M_2)$ be the element corresponding to ϕ . We put $\text{tr}(\phi) := i_\iota(\phi')$, where $\iota \in M^\vee \otimes_{A^e} M$ is the canonical element.

5.6. The differential operator D_∇ . We are now prepared to define the map D_∇ . Consider the canonical element ι from Definition 4.3.6. Let us now consider ι as an element of $\text{Der}(A)^\vee \otimes_{A^e} (\Omega^1 A)^\vee \subset \mathcal{F}_0(T_A(\text{Der}(A) \oplus \Omega^1 A)^\vee)$. We then have the obtained contraction i_ι as in Proposition 5.5.7. Gradings are as in Section 4.1.

Definition 5.6.1. We define an operator $D_\nabla : \mathcal{F}_m(T_A \text{Der}(A)) \rightarrow \mathcal{F}_m(T_A \text{Der}(A))$ by $D_\nabla = i_\iota \circ \nabla$.

The map D_∇ satisfies BV-like identities involving the double Poisson bracket, analogous to (5.1.5). We have the following main result (some notation will be explained after the statement).

Theorem 5.6.2. *Let ∇ be any torsion-free bimodule connection.*

- (i) *D_∇ is a differential operator of order ≤ 2 and degree -1 (for $|A| = 1, |\text{Der}(A)| = 1$) on $\mathcal{F}(T_A \text{Der}(A))$, commuting with wheeled contractions, whose principal symbol $\Gamma_2(D_\nabla)$ is \pm the Schouten-Nijenhuis bracket. That is, one has the BV identity, for homogeneous $\xi, \eta \in \mathcal{F}(T_A \text{Der}(A))$,*

$$(-1)^{|\xi|+1} \{\xi, \eta\} = D_\nabla(\xi \otimes \eta) - D_\nabla(\xi) \otimes \eta - (-1)^{|\xi|} \xi \otimes D_\nabla(\eta). \quad (5.6.3)$$

- (ii) *The operator $\nabla^2 : \text{Der}(A) \rightarrow \mathcal{F}_1(T_A(\text{Der}(A) \oplus \Omega^1 A))$ is A^e -linear, and one has*

$$D_\nabla^2 = i_{\text{tr}(\nabla^2)}. \quad (5.6.4)$$

- (iii) *More generally, if we adjoin formally the element $(1 + \text{rk}(\mathbb{D}\text{er}(A)))^{-1}$ to $\mathcal{F}(A)$ (for $1 \in \mathcal{F}_0(A)$ the wheelgebra unit, not the unit element of A), then the above yields a one-to-one correspondence between generalized torsion-free bimodule connections $\nabla = (\nabla_\ell, \nabla_r)$ on $\mathbb{D}\text{er}(A)$ and differential operators $D \in \mathcal{D}_0(A)_{\leq 2}[-2]$ whose principal symbol is the Schouten-Nijenhuis bracket. Under this correspondence, the trace-flat connections are the ones which map to wheeled BV operators. Two torsion-free generalized connections ∇, ∇' map to the same differential operator $D_\nabla = D_{\nabla'}$ if and only if $\text{div } \nabla = \text{div } \nabla'$.*

Here, $\text{tr}(\nabla^2)$ is defined, as in Definition 5.5.9, by contracting the input with the $\mathbb{D}\text{er}(A)$ in the output, but viewing the output $\Omega^1 A$'s as separate from $\mathbb{D}\text{er}(A)$ (like M_1, M_2 in Definition 5.5.9). The notation $\mathcal{D}_0(A)_{\leq 2}[-2]$ means sending $\mathcal{F}_m(A)$ to $\mathcal{F}_m(A)$, having order ≤ 2 , and degree -2 using $|\mathbb{D}\text{er}(A)| = 1$, $A=0$.

A *generalized connection* $\nabla = (\nabla_\ell, \nabla_r)$ is a pair of maps $\nabla_\ell, \nabla_r : \mathbb{D}\text{er}(A) \rightarrow \Omega^1 A \otimes_{A^e} \mathcal{W}\mathbb{D}\text{er}(A)_2$, such that

$$\nabla_\ell(a\xi) = a \otimes_A \nabla_\ell(\xi) + da \otimes_A \xi, \quad \nabla_\ell(\xi a) = \nabla_\ell(\xi) \otimes_A a, \quad (5.6.5)$$

$$\nabla_r(a\xi) = a \otimes_A \nabla_r(\xi), \quad \nabla_r(\xi a) = \nabla_r(\xi) \otimes_A a + \xi \otimes_A da. \quad (5.6.6)$$

One may define the torsion exactly as in Section 4.3, and hence the torsion-free condition.

By formally adjoining the element $(1 + \text{rk}(\mathbb{D}\text{er}(A)))^{-1}$, we mean to consider polynomials in this element (considered to have wheeled degree zero) with coefficients in $\mathcal{F}(A), \mathcal{D}(A)$; the element $\text{rk}(\mathbb{D}\text{er}(A)) \in \text{Sym}^2(A/[A, A]) \subset \mathcal{F}_0(A)$ is $\text{rk}(\mathbb{D}\text{er}(A)) := \pi(i_\iota(\iota))$, where $\pi : (A/[A, A])^{\otimes 2} \twoheadrightarrow \text{Sym}^2(A/[A, A])$ is the projection.⁴ Finally, $\text{div } \nabla := i_\iota \circ \nabla$.

Corollary 5.6.7. *If ∇ is torsion-free, then D_∇ endows $\mathcal{F}(T_A \mathbb{D}\text{er}(A))$ with the structure of a BV wheelgebra if and only if $\text{tr}(\nabla^2) = 0$. The induced wheeled Gerstenhaber structure is the Schouten-Nijenhuis one.*

In particular, this motivates the definition

Definition 5.6.8. An associative algebra A is wheeled Calabi-Yau if A has a trace-flat torsion-free bimodule connection.

By the above, if A is wheeled Calabi-Yau, then $\mathcal{F}(T_A \mathbb{D}\text{er}(A))$ is equipped with a wheeled BV structure extending the Schouten-Nijenhuis wheeled Gerstenhaber structure. (Also, by the above remarks, the two notions are not that different).

Corollary 5.6.9. *Suppose ∇ is a torsion-free connection on $\mathbb{D}\text{er}(A)$. Then, D_∇ induces maps*

$$D_\nabla : T_A \mathbb{D}\text{er}(A) \otimes T_A \mathbb{D}\text{er}(A) \longrightarrow T_A \mathbb{D}\text{er}(A) \otimes T_A \mathbb{D}\text{er}(A) \otimes \text{Sym}^{\leq 1} A_{\text{cyc}}, \quad \text{and} \\ T_A \mathbb{D}\text{er}(A) \longrightarrow T_A \mathbb{D}\text{er}(A) \otimes (T_A \mathbb{D}\text{er}(A))_{\text{cyc}},$$

⁴Note that the element $i_\iota(\iota) \in (A/[A, A])^{\otimes 2}$ itself is already canonical, and under the representation functor, it is taken to the dimension of the representation varieties.

which satisfy the identities, for $\xi, \eta \in T_A \mathbb{D}er(A)$,

$$D_{\nabla}(\xi \otimes \eta) - D_{\nabla}(\xi) \otimes \eta - (-1)^{|\xi|} \xi \otimes D_{\nabla}(\eta) = (-1)^{|\xi|+1} \{\xi, \eta\}, \quad (5.6.10)$$

$$D_{\nabla}(\xi \eta) - D_{\nabla}(\xi) \eta - (-1)^{|\xi|} \xi D_{\nabla}(\eta) = (-1)^{|\xi|+1} (\text{pr} \otimes 1) \{\xi, \eta\}, \quad (5.6.11)$$

where $\text{pr} : T_A \mathbb{D}er(A) \twoheadrightarrow (T_A \mathbb{D}er(A))_{\text{cyc}}$ is the projection.

Proof of Corollary 5.6.9. We will prove the corollary independently of the theorem, to help explain what is going on in a simpler setting. Recall that $D_{\nabla} = i_{\iota} \circ \nabla$, and the contraction i_{ι} with the canonical element is, essentially by definition, a signed sum over ways of contracting a copy of $\mathbb{D}er(A)$ with $\Omega^1 A$. Hence, if $\xi = \xi_1 \otimes_A \cdots \otimes_A \xi_m$ and $\eta = \eta_1 \otimes_A \eta_2 \otimes_A \cdots \otimes_A \eta_n$, the LHS of (5.6.10) can be expanded as

$$\begin{aligned} & \sum_{i=1}^m \sum_{j=1}^n \pm \left((\eta_1 \otimes_A \cdots \otimes_A \eta_{j-1} \otimes_A (((\nabla_{\ell})_{\xi_i} \eta_j)' - ((\nabla_r)_{\eta_j} \xi_i)')) \otimes_A \xi_{i+1} \otimes_A \cdots \otimes_A \xi_m \right) \\ & \otimes (\xi_1 \otimes_A \cdots \otimes_A \xi_{i-1} \otimes_A (((\nabla_{\ell})_{\xi_i} \eta_j)'' - ((\nabla_r)_{\eta_j} \xi_i)')) \otimes_A \eta_{j+1} \otimes_A \cdots \otimes_A \eta_n \\ & + \pm (\eta_1 \otimes_A \cdots \otimes_A \eta_{j-1} \otimes_A (((\nabla_r)_{\xi_i} \eta_j)' - ((\nabla_{\ell})_{\eta_j} \xi_i)')) \otimes_A \xi_{i+1} \otimes_A \cdots \otimes_A \xi_m \\ & \otimes (\xi_1 \otimes_A \cdots \otimes_A \xi_{i-1} \otimes_A (((\nabla_r)_{\xi_i} \eta_j)'' - ((\nabla_{\ell})_{\eta_j} \xi_i)')) \otimes_A \eta_{j+1} \otimes_A \cdots \otimes_A \eta_n \Big), \end{aligned} \quad (5.6.12)$$

where above we use the restriction/quotient described in the statement of the proposition, and where signs are determined in the appropriate way from the superbraidings. By (4.5.3) (or (4.5.4) in the case m is odd), since ∇ is torsion-free, (5.6.10) follows from the graded double Poisson condition for Van den Bergh's Schouten-Nijenhuis bracket $\{\{-, -\}\}$ (or, equivalently, the wheeled Poisson condition for $\{-, -\}$), from which we deduce (5.6.11) using appropriate wheeled contractions. \square

5.7. The quiver case. Before proving the theorem, we explain what it says in the case of quivers. Let $A = P_Q$, the path algebra of a quiver Q . In this case, it is well known ([BLB02, Gin01]) that $(P_Q)_{\text{cyc}}$ is a Lie algebra, so one may consider its universal enveloping algebra, which quantizes $\text{Sym } P_Q \cong \mathcal{F}_0(T_A \mathbb{D}er(A))$. However, this is *not* the quantization $\mathcal{D}_0(\mathcal{F}(T_A \mathbb{D}er(A)))$: this is a quantization of the *Lie bialgebra* structure defined in [Sch05], called the *quantized necklace algebra*. So, in a sense, taking differential operators performs two quantizations at once: the universal enveloping, followed by the quantized universal enveloping.

- Theorem 5.7.1.** (i) *A is wheeled Calabi-Yau, equipped with the trivial connection on $\mathbb{D}er(A)$, whose associated wheeled BV structure incorporates the double/necklace Lie (co)bracket as in (1.4.3).*
- (ii) *The degree-zero part $\mathcal{D}_0(\mathcal{F}(P_Q))$ of wheeled differential operators on P_Q is isomorphic to the quantized necklace algebra from [Sch05] (for $\hbar = 1$).*

- (iii) The action of $\mathcal{D}_0(\mathcal{F}(P_Q))$ on $\mathcal{F}_0(P_Q)$ is the limit, as $\mathbf{d} \rightarrow \infty$, of the representations of the quantized necklace algebra as differential operators on $\text{Rep}_{\mathbf{d}}(Q)$ (for $\hbar = 1$).
- (iv) One has a wheeled analogue of the infinitesimal Weil representation: the quadratic-in-path-length subspace of $(P_{\overline{Q}})_{\text{cyc}}$ is a Lie subalgebra of $\mathcal{D}(\mathcal{F}(P_Q))$ isomorphic to $\mathfrak{sp}(Q)$, which acts via a wheeled infinitesimal Weil representation on $\mathcal{F}(P_Q)$.

Note that, in particular, the degree-zero part of a BV wheelgebra is an ordinary BV algebra (indeed, the degree-zero part of any type of wheelgebra is an ordinary algebra of that type), and so the algebra $\mathcal{F}_0(P_{\overline{Q}}) = \text{SuperSym}(P_{\overline{Q}})_{\text{cyc}}$, in particular obtains an ordinary BV algebra structure. This coincides with the one obtained from the construction of §1.4, viewing $(P_{\overline{Q}})_{\text{cyc}}$ as an involutive Lie bialgebra.

Example 5.7.2. In the case that the quiver has one vertex, $A = TV$ is merely a tensor algebra on a vector space V . In this case, the BV algebra $\text{SuperSym}(TV)_{\text{cyc}}$ coincides with the BV algebra F considered in [Bar07], [Bar09, §1] (cf. §1.5), although there V is allowed to be a supervector space rather than merely a vector space (we could also work in this generality, or in the quiver setting, we could allow edges to be even or odd).

Proof. (i) The algebra $T_A \mathbb{D}\text{er}(A)$, as was noticed in [VdB08], is $P_{\overline{Q}}$, equipped with the grading $|Q| = 0, |Q^*| = 1$. There is a canonical bimodule connection ∇ on $\mathbb{D}\text{er}(A)$ given by $\nabla(e^*) = 0$ for all $e^* \in Q^*$, called the *trivial connection*. Using this connection, the operator D_{∇} acts as in (1.4.3). It is immediate that $\nabla^2 = 0$ (since $\nabla(e^*) = 0$ for all $e^* \in Q^*$, and ∇^2 is A^e -linear). Hence, it follows that P_Q is wheeled Calabi-Yau, and that the operator D_{∇} is wheeled BV.

(ii) This follows because $\mathbb{D}\text{er}(A)$ is a free bimodule generated by ∂_e for $e \in Q$, using Theorem 3.6.7 and the description of the quantized necklace algebra from [Sch05].

(iii) This follows from asymptotic bijectivity ([Sch05, GS06, EG07a]) of the representation functor $\mathcal{F}_0(P_{\overline{Q}}) \rightarrow \text{Rep}_{\mathbf{d}}(\overline{Q})^{\text{GL}_{\mathbf{d}}}$, together with Theorem 3.6.7 and the fact that $\mathcal{F}(P_{\overline{Q}}) \cong \mathcal{F}(T_{P_Q} \mathbb{D}\text{er}(P_Q))$ (using that $\mathbb{D}\text{er}(A)$ is free as in (ii)).

(iv) We claim that the map $[P_{\overline{Q}}[2]]_{\text{cyc}} \hookrightarrow \mathcal{D}_0(\mathcal{F}(A))$ given by symmetrization,

$$[ee^*] \mapsto \frac{1}{2}([e \circ \partial_e]_{\text{cyc}} + [\partial_e \circ e]_{\text{cyc}}), \quad e \in Q, \quad (5.7.3)$$

is an embedding of Lie algebras (using the Lie structure on $[P_{\overline{Q}}[2]]_{\text{cyc}}$ obtained from the symplectic pairing of $\langle \overline{Q} \rangle$), where $e \circ \partial_e$ says to first perform ∂_e and then multiply by e (on the input and output corresponding to ∂_e), and similarly for $\partial_e \circ e$. All other elements $[ef], [e^*f^*]$ for all $e, f \in Q$, and $[ef^*]$ for $f \neq e$, can be mapped by the same symmetrization procedure, but we don't need to symmetrize since e, f twisted-commute (and similarly ∂_e, ∂_f and e, ∂_f). This claim is easy to verify explicitly, and it is all we need. \square

5.8. Proof of Proposition 5.1.6. In order to prove the theorem, we first need to complete the proof of Koszul's Proposition 5.1.6, as promised.

We prove a more general result than Proposition 5.1.6. Namely, we give a proof that works in the algebraic setting, as well as the smooth or complex settings. It is enough to assume that our variety X is affine, so that \mathcal{O}_X , T_X , and Ω_X are generated by local sections, since the statements (particularly (5.1.7)) are local.

Now, assuming that X is smooth affine and finite-dimensional, we give a global argument, that does not rely on further localization (we will be very careful about this at every step). Let $A = \Gamma(X, \mathcal{O}_X)$, $\text{Der } A = \Gamma(X, T_X)$, and $\Omega^1 A = \Gamma(X, \Omega_X)$. Since X is smooth, $\text{Der } A$ and $\Omega^1 A$ are finitely-generated projective. Also, $\Omega^1 A$ and $\text{Der } A$ are projectively dual (i.e., $\Omega^1 A \cong \text{Hom}_A(\text{Der } A, A)$ and $\text{Der } A \cong \text{Hom}_A(\Omega^1 A, A)$).

Let us explicitly write the canonical element $\iota \in \text{Der } A \otimes_A \Omega^1 A$ corresponding to the identity as

$$\iota = \sum_{i=1}^n \xi_i \otimes_A \omega^i, \quad (5.8.1)$$

for some $\xi_i \in \text{Der } A$, $\omega^i \in \Omega^1 A$. (NOTE: the “ i ” appearing in ω^i is a superscript, *not* an exponent.)

Let ∇ be a connection on T_X which is torsion-free. Explicitly,

$$\nabla_{\xi_i} \xi_j - \nabla_{\xi_j} \xi_i = [\xi_i, \xi_j], \quad \forall i, j. \quad (5.8.2)$$

Notation 5.8.3. Let \circ denote the action of vector fields on functions: $\xi \circ f := \xi \lrcorner df$ for all functions $f \in A$ and vector fields $\xi \in \text{Der } A$.

We now wish to verify explicitly the identity

$$D_{\nabla}^2 = -i_{\text{tr}(\nabla^2)}. \quad (5.8.4)$$

Since we already know (as in Section 5.1) that the LHS is a differential operator of order ≤ 2 and degree -2 , it must be given by contraction with some two-form. To verify that the above formula holds, it suffices to compute both sides applied to an arbitrary element of the form $\xi \wedge \eta$, for $\xi, \eta \in \text{Der } A$.

Notation 5.8.5. For any vector $\eta \in \text{Der } A$, let us define coefficients $\eta^i \in A$ by $\eta^i := \eta \lrcorner \omega^i$ (when there is no possible confusion). In particular, this implies that $\eta = \sum_i \eta^i \xi_i$.

Whenever we have a tensor $a \in V \otimes W$, we write $a = a' \otimes a''$, using Sweedler's notation (4.5.11).

Finally, note that one has the well known formula for ∇^2 , which we prove as a warmup (and because we will need to prove this in the wheeled setting):

Lemma 5.8.6.

$$(\xi \wedge \eta) \lrcorner \nabla^2 = \nabla_{\xi} \nabla_{\eta} - \nabla_{\eta} \nabla_{\xi} - \nabla_{[\xi, \eta]}. \quad (5.8.7)$$

Proof. We have

$$\nabla(\theta) = \sum_i \nabla_{\xi_i} \theta \otimes_A \omega^i. \quad (5.8.8)$$

Then, we have

$$\nabla^2(\theta) = \sum_{i,j} \nabla_{\xi_i} \nabla_{\xi_j} \theta \otimes_A (\omega^i \wedge \omega^j) + \sum_i \nabla_{\xi_i} \theta \otimes_A d\omega^i. \quad (5.8.9)$$

We need to use the following formula (which we note can be proved by writing $\omega = a db$ without localizing):

$$d\omega \lrcorner (\xi \wedge \eta) = \xi \circ (\omega \lrcorner \eta) - \eta \circ (\omega \lrcorner \xi) - [\xi, \eta] \lrcorner \omega. \quad (5.8.10)$$

Thus, we may expand (5.8.9) applied to $\xi \wedge \eta$, for each of the summations on the RHS separately, as

$$\begin{aligned} (\xi \wedge \eta) \lrcorner \sum_{i,j} \nabla_{\xi_i} \nabla_{\xi_j} \theta (\omega^i \wedge \omega^j) &= \sum_{i,j} (\xi^i \eta^j - \xi^j \eta^i) \nabla_{\xi_i} \nabla_{\xi_j} \theta \\ &= \sum_j \eta^j \nabla_{\xi} \nabla_{\xi_j} \theta - \xi^j \nabla_{\eta} \nabla_{\xi_j} \theta \\ &= (\nabla_{\xi} \nabla_{\eta} - \nabla_{\eta} \nabla_{\xi}) \theta - ((\xi \circ \eta^j) - (\eta \circ \xi^j)) \nabla_{\xi_j} \theta. \end{aligned} \quad (5.8.11)$$

$$\begin{aligned} \sum_i \nabla_{\xi_i} \theta \otimes_A d\omega_i \lrcorner (\xi \wedge \eta) &= \sum_i ((\xi \circ \eta^i) - (\eta \circ \xi^i) - [\xi, \eta]^i) \nabla_{\xi_i} \theta \\ &= -\nabla_{[\xi, \eta]} \theta + \sum_i ((\xi \circ \eta^i) - (\eta \circ \xi^i)) \nabla_{\xi_i} \theta. \end{aligned} \quad (5.8.12)$$

Summing (5.8.11) and (5.8.12), we deduce the desired identity. \square

We will need the formula

$$\mathrm{tr}(F) = \sum_i F(\xi_i)^i, \quad (5.8.13)$$

for any endomorphism $F : \mathrm{Der} A \rightarrow \mathrm{Der} A$. It is immediate that this is the same as applying the contraction i_{ι} to F considered as an element of $\mathrm{Der} A \otimes_A \Omega^1 A$.

Now, we proceed to the main

Lemma 5.8.14.

$$D_{\nabla}^2(\xi \wedge \eta) = -\mathrm{tr}(\nabla_{\xi} \nabla_{\eta} - \nabla_{\eta} \nabla_{\xi} - \nabla_{[\xi, \eta]}). \quad (5.8.15)$$

Proof. First, by applying D_{∇} twice, and canceling the $\pm D_{\nabla}(\xi) D_{\nabla}(\eta)$ terms, we obtain

$$D_{\nabla}^2(\xi \wedge \eta) = \eta \circ D_{\nabla}(\xi) - \xi \circ D_{\nabla}(\eta) + D_{\nabla}[\xi, \eta]. \quad (5.8.16)$$

By definition, the RHS expresses in terms of generators as

$$\sum_i \eta \circ (\nabla_{\xi_i}(\xi)^i) - \xi \circ (\nabla_{\xi_i}(\eta)^i) + \nabla_{\xi_i}([\xi, \eta])^i. \quad (5.8.17)$$

Next, we apply torsion-freeness to rewrite the RHS as

$$\sum_i \eta \circ (\nabla_\xi(\xi_i)^i - [\xi, \xi_i]) - \xi \circ (\nabla_\eta(\xi_i)^i - [\eta, \xi_i]) + \nabla_{[\xi, \eta]}(\xi_i)^i - [[\xi, \eta], \xi_i]^i. \quad (5.8.18)$$

Furthermore, we note that

$$\begin{aligned} \nabla_\xi \nabla_\eta(\xi_i)^i &= \sum_\ell \nabla_\xi(\nabla_\eta(\xi_i)^\ell \xi_\ell)^i = \sum_\ell \xi \circ (\nabla_\eta(\xi_i)^\ell)(\xi_\ell \lrcorner \omega^i) + \nabla_\xi(\xi_\ell)^i \nabla_\eta(\xi_i)^\ell \\ &= \sum_\ell \xi \circ ((\xi_\ell \lrcorner \omega^i) \nabla_\eta(\xi_i)^\ell) - \xi \circ (\xi_\ell \lrcorner \omega^i) \nabla_\eta(\xi_i)^\ell + \nabla_\xi(\xi_\ell)^i \nabla_\eta(\xi_i)^\ell \\ &= \xi \circ (\nabla_\eta(\xi_i)^i) + \sum_\ell (\nabla_\xi(\xi_\ell)^i - \xi \circ (\xi_\ell \lrcorner \omega^i)) \nabla_\eta(\xi_i)^\ell. \end{aligned} \quad (5.8.19)$$

Substituting this formula (and the same with ξ, η swapped) into (5.8.18), we obtain

$$\begin{aligned} D_\nabla^2(\xi \wedge \eta) + \text{tr}(\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi, \eta]}) \\ = \sum_i -\eta \circ [\xi, \xi_i] + \xi \circ [\eta, \xi_i] - [[\xi, \eta], \xi_i]^i \\ + \sum_{i, \ell} (\eta \circ (\xi_\ell \lrcorner \omega^i)) \nabla_\xi(\xi_i)^\ell - \xi \circ (\xi_\ell \lrcorner \omega^i) \nabla_\eta(\xi_i)^\ell. \end{aligned} \quad (5.8.20)$$

It remains to prove that the RHS is zero. Using the Jacobi identity, we have

$$\begin{aligned} [[\xi, \eta], \xi_i]^i &= [\xi, [\eta, \xi_i]]^i - [\eta, [\xi, \xi_i]]^i = \sum_\ell [\xi, [\eta, \xi_i]^\ell \xi_\ell]^i - [\eta, [\xi, \xi_i]^\ell \xi_\ell]^i \\ &= \xi \circ ([\eta, \xi_i]^\ell)(\xi_\ell \lrcorner \omega^i) - \eta \circ ([\xi, \xi_i]^\ell)(\xi_\ell \lrcorner \omega^i) + \sum_\ell [\xi, \xi_\ell]^i [\eta, \xi_i]^\ell - [\eta, \xi_\ell]^i [\xi, \xi_i]^\ell \\ &= \xi \circ [\eta, \xi_i]^i - \eta \circ [\xi, \xi_i]^i + \sum_\ell -\xi \circ (\xi_\ell \lrcorner \omega^i) [\eta, \xi_i]^\ell + \eta \circ (\xi_\ell \lrcorner \omega^i) [\xi, \xi_i]^\ell. \end{aligned} \quad (5.8.21)$$

Thus, combining (5.8.20) and (5.8.21), and applying torsion-freeness, we need only show that

$$\sum_{i, \ell} (\eta \circ (\xi_\ell \lrcorner \omega^i)) \nabla_{\xi_i}(\xi)^\ell - \xi \circ (\xi_\ell \lrcorner \omega^i) \nabla_{\xi_i}(\eta)^\ell = 0. \quad (5.8.22)$$

More generally, we claim that

$$\sum_{i, \ell} d(\xi_\ell \lrcorner \omega^i) \otimes_A \xi_i \otimes_A \omega^\ell = 0. \quad (5.8.23)$$

To show this, let us define

$$M_{ij} := \xi_i \lrcorner \omega^j, \quad (5.8.24)$$

and contract the second and third components of the LHS (5.8.23) with $\omega^j \otimes_A \xi_k$. We obtain

$$\sum_{i, \ell} d(M_{\ell i}) M_{ij} M_{k\ell}. \quad (5.8.25)$$

To show that this is zero (for all j, k), first note that $\xi_i = \sum_j (\xi_i \lrcorner \omega^j) \xi_j$ implies

$$\sum_j M_{ij} M_{jk} = M_{ik}. \quad (5.8.26)$$

Applying (5.8.26) together with the Leibniz rule (i.e., $d(M_{\ell i}) M_{ij} = d(M_{\ell i} M_{ij}) - M_{\ell i} d(M_{ij})$), we obtain

$$\sum_{i, \ell} d(M_{\ell i}) M_{ij} M_{k\ell} = \sum_{\ell} d(M_{\ell j}) M_{k\ell} - \sum_i M_{ki} d(M_{ij}) = 0. \quad (5.8.27)$$

This concludes the proof. \square

5.9. Proof of Theorem 5.6.2. The difficulty is in proving part (ii), which we do first, using part (i). For this, the proof in the previous subsection applies to our setting as well, provided we make the following interpretations and conventions: First, ξ, η are now double derivations. We use (4.5.1): $\nabla_{\xi} := (\nabla_{\ell})_{\xi} + (\nabla_r)_{\xi}$. Secondly, in computing $D_{\nabla}^2(\xi \otimes \eta)$, we can afford to lose track of (signed) permutations of tensor components, if we apply, at the end, whatever signed wheeled permutation is necessary so as to have the substitution $\xi \mapsto a\xi b$ apply left-multiplication by a under the first A -module structure, and similarly apply right-multiplication by b to the first A^{op} -module structure. Here we use that each $\mathcal{F}_m(T_A M)$ is not merely an S_m -module, but an $S_m \times S_m$ -module, given by permuting the left and right components separately.

Furthermore, the tensor products over A must become tensor products over A^e , and \wedge becomes \otimes_A . Then \lrcorner becomes the total contraction. Finally, in the arguments about M_{ij} to prove (5.8.23), the Leibniz rule still applies, but now $M_{ij} \in A \otimes A$, and the multiplication is A^e -multiplication: if $\xi_i \lrcorner \omega^j$ takes the bimodule action on ω^j to outer action, as in (4.5.13), then $M_{ij} M_{kl}$ glues the outer action on M_{ij} to the inner on M_{kl} (i.e., puts M_{ij} inside M_{kl} and multiplies).

Making all of these changes, the proof of (ii) goes through.

(i) Since $D_{\nabla} = i_{\ell} \circ \nabla$, the definitions show that we are summing over applying operations to two terms in a tensor product. This shows that D_{∇} must have order ≤ 2 . Also, the number of $\mathbb{D}\text{er}(A)$'s (the degree) goes down by one. Then, we apply Theorem 3.6.7 and Proposition 4.5.2.

(iii) It is easy to see that a generalized connection is torsion-free if and only if it satisfies the BV identity (5.6.3) (we say it “generates the S-N bracket”).

We first show that any differential operator ϕ that generates the S-N bracket is of the form $\phi = D_{\nabla}$ for an appropriate torsion-free generalized bimodule connection $\nabla = (\nabla_{\ell}, \nabla_r)$. To see this, first let ∇' be **any** torsion-free bimodule connection (which exists because we can let ∇'_{ℓ} be arbitrary, and then ∇'_r is determined by (4.5.3)). Then, $\Gamma_2(\phi - D_{\nabla'}) = 0$, so $\phi - D_{\nabla'}$ is a differential operator of order one and degree -1 . Such a map is the same as an A^e -linear map $\mathbb{D}\text{er}(A) \rightarrow \mathcal{F}_1(A)$, i.e., contraction with an element $\alpha \in \mathcal{F}_0(\Omega^1 A)[-1]$. It suffices to show that any such element can

be realized as $D_{\nabla} - D_{\nabla'}$ for some connection ∇ . Finding such a ∇ is the same as finding $\nabla_{\ell} - \nabla'_{\ell} \in \mathcal{W}\mathrm{Der}(A)_2 \otimes_{A^e \otimes A^e} (\Omega^1 A \otimes (\Omega^1 A)')$ (where $(\Omega^1 A)'$ is a distinctly-labeled copy of $\Omega^1 A$), which uniquely determines ∇_r by the torsion-free condition. Precisely, $\nabla_r - \nabla_{r'}$ swaps $\Omega^1 A$ with $(\Omega^1 A)'$.

Next, let us write $\alpha = \sum_i [\alpha_i]_{\mathrm{cyc}} \otimes X_i$ for some $\alpha_i \in \Omega^1 A$, and some $X_i \in \mathcal{F}_0(A)$. It suffices to find an element $Y \in \mathcal{W}\mathrm{Der}(A)_{2 \otimes A^e \otimes A^e}(\Omega^1 A \otimes \Omega^1 A)$ such that

$$i_{\iota}(Y) = [\alpha_i]_{\mathrm{cyc}}. \quad (5.9.1)$$

There is a natural choice, $Y = [\alpha_i]_{\mathrm{cyc}} \otimes \iota$. We compute

$$i_{\iota}(Y) = [\alpha_i]_{\mathrm{cyc}} + [\alpha_i]_{\mathrm{cyc}} \otimes \mathrm{rk}(\mathrm{Der}). \quad (5.9.2)$$

Thus, if we are allowed to divide by $(1 + \mathrm{rk}(\mathrm{Der}))$ (where $1 \in \mathcal{F}_0(T_A \mathrm{Der}(A))$ is the unit of the wheelgebra, not of A), then we can produce such a ∇ .

Now, a straightforward generalization of (ii) shows that the operator D_{∇} associated to a torsion-free generalized bimodule connection ∇ satisfies $D_{\nabla}^2 = 0$ if and only if $\mathrm{tr}(\nabla^2) = 0$: in fact, (5.6.4) generalizes to this setting.

6. The representation functor

6.1. Main constructions. If A is a finitely-generated associative algebra and V is a finite-dimensional vector space, one can consider the affine variety $\mathrm{Rep}_V A$ of representations of A in V . In this section we sketch how wheeled constructions on A correspond to the usual notions on $\mathrm{Rep}_V A$: for instance, a wheeled differential operator on A gives rise to a differential operator on $\mathbf{k}[\mathrm{Rep}_V A]$ for every V .

Throughout, A will denote an associative algebra over \mathbf{k} and V a finite-dimensional vector space over \mathbf{k} . Let $\mathrm{Hom}_{\mathbf{k}\text{-alg}}(X, Y)$ denote the space of \mathbf{k} -algebra homomorphisms from X to Y . Recall from, e.g., [Gin05, §12] the

Definition 6.1.1. The affine \mathbf{k} -scheme $\mathrm{Rep}_V(A)$ is defined as the scheme representing the functor $B \mapsto \mathrm{Hom}_{\mathbf{k}\text{-alg}}(A, B \otimes \mathrm{End}(V))$.

Next, we define from this a certain commutative wheelgebra. First, we define the *endomorphism wheelgebra* of V (which is analogous to its endomorphism operad, (wheeled) PROP, etc.):

Definition 6.1.2. Let $\mathrm{WEnd}(V) := \bigoplus_m \mathrm{End}(V)^{\otimes m}$ be the endomorphism wheelgebra of V , where the $S_m \times S_m$ -module structure is the natural one on $(V^*)^{\otimes m} \otimes V^{\otimes m}$ by permuting components separately in V and V^* , and the contraction operation is given by the trace pairings $\mathrm{tr}_{i,j}$ of the i -th component of V^* with the j -th component of V .

Definition 6.1.3. Given a (commutative, Lie, Poisson, etc.) wheelgebra \mathcal{W} , and a commutative algebra B , let $\mathcal{W} \otimes B$ denote the (commutative, Lie, Poisson, etc.) wheelgebra $(\mathcal{W} \otimes B)(m) := \mathcal{W}(m) \otimes B$, with all structure maps given by tensoring by Id_B .

The following will be our main object of study:

Definition 6.1.4. The *representation wheelgebra* $\mathrm{WRep}_V(A)$ is defined as

$$\mathrm{WRep}_V(A) := \mathbf{k}[\mathrm{Rep}_V(A)] \otimes \mathrm{WEnd}(V). \quad (6.1.5)$$

This is equipped with canonical evaluation map $\mathrm{ev} : \mathcal{F}(A) \rightarrow \mathrm{WRep}_V(A)$, defined as the unique extension of the standard evaluation map $\mathrm{ev} : A \rightarrow \mathbf{k}[\mathrm{Rep}_V(A)] \otimes \mathrm{End}(V)$ to a morphism of wheelgebras. Recall from, e.g., [Gin05, §12], that the standard evaluation map is the universal map such that any algebra map $\rho : A \rightarrow B \otimes \mathrm{End}(V)$ factors through ev by a morphism of algebras $B \rightarrow \mathbf{k}[\mathrm{Rep}_V(A)]$. Likewise, its extension has the universal property that any morphism of wheelgebras $\mathcal{F}(A) \rightarrow B \otimes \mathrm{WEnd}(V)$ factors through $\mathrm{WRep}_V(A)$ via a morphism of the form $(\phi \otimes \mathrm{Id}) : B \otimes \mathrm{WEnd}(V) \rightarrow \mathbf{k}[\mathrm{Rep}_V(A)] \otimes \mathrm{WEnd}(V)$, where $\phi : B \rightarrow \mathbf{k}[\mathrm{Rep}_V(A)]$ is an algebra morphism. The image of the evaluation maps is the space of $\mathrm{GL}(V)$ -invariants: it is clear that the image is contained in the space of invariants. For surjectivity, we argue similarly to [LBP90]: as $\mathrm{GL}(V)$ -representations, $\mathbf{k}[\mathrm{Rep}_V(A)]$ is generated by $(A \otimes \mathrm{End}(V))$ under the map $A \otimes T \mapsto \mathrm{tr}(\mathrm{ev}(a) \cdot T)$, which is a direct sum of copies of $V \otimes V^*$. On the other hand, $\mathrm{End}(V)^{\otimes m} = V^{\otimes m} \otimes (V^*)^{\otimes m}$, and the fundamental theorem of invariant theory implies that all $\mathrm{GL}(V)$ -invariants are spanned by complete contractions (matchings of the V components with the V^* components using the canonical pairing).

In fact, the evaluation morphism of wheelgebras gives a natural presentation of $\mathbf{k}[\mathrm{Rep}_V(A)]$: see (6.1.10) below.

This motivates the following more general definition:

Definition 6.1.6. Given any wheelgebra \mathcal{W} and vector space V , let $(\mathrm{RA}_V(\mathcal{W}), \mathrm{ev})$ denote the universal associative algebra together with evaluation morphism $\mathrm{ev} : \mathcal{W} \rightarrow \mathrm{RA}_V(\mathcal{W}) \otimes \mathrm{WEnd}(V)$, such that any morphism of wheelgebras $\mathcal{W} \rightarrow B \otimes \mathrm{WEnd}(V)$ factors through ev via an algebra morphism $B \rightarrow \mathrm{RA}_V(\mathcal{W})$. It is called the *representation algebra* of \mathcal{W} . Similarly, let $\mathrm{WRep}(\mathcal{W}) := \mathrm{RA}_V(\mathcal{W}) \otimes \mathrm{WEnd}(V)$.

In the case of ordinary algebras, instead of wheelgebras, this definition is equivalent to Definition 6.1.1 for $\mathbf{k}[\mathrm{Rep}_V(A)]$: the latter is the universal associative algebra equipped with a morphism $A \rightarrow \mathbf{k}[\mathrm{Rep}_V(A)] \otimes \mathrm{End}(V)$ such that any morphism $A \rightarrow B \otimes \mathrm{End}(V)$ factors through ev via an algebra morphism $B \rightarrow \mathbf{k}[\mathrm{Rep}_V(A)]$.

Definition 6.1.6 relies on

Theorem 6.1.7. *For all wheelgebras, $\mathrm{RA}_V(\mathcal{W})$ exists. It is equipped with an action of $\mathrm{GL}(V)$, and the image of the evaluation morphism is $\mathrm{WRep}_V(\mathcal{W})^{\mathrm{GL}(V)}$. Finally, if \mathcal{W} is (almost) commutative, Poisson, or BV, so is $\mathrm{RA}_V(\mathcal{W})$.*

The proof is postponed to the end of the section, since the considerations are a generalization of the following theorem in the case of $\mathcal{W} = \mathcal{F}(A)$ (which we prove independently of the theorem):

Theorem 6.1.8. *Let A be an associative algebra and V a vector space.*

- (i) *The representation algebra $\mathrm{WRep}_V(\mathcal{F}(A))$ coincides with $\mathrm{WRep}_V(A)$.*

- (ii) *There is a canonical evaluation morphism of filtered wheelgebras, $\text{ev} : \mathcal{D}(\mathcal{F}(A)) \rightarrow (D(\text{Rep}_V(A)) \otimes \text{WEnd}(V))$.*
- (iii) *If A is smooth, then $\text{RA}_V(\mathcal{D}(\mathcal{F}(A))) = D(\text{Rep}_V(A))$, the algebra of differential operators on the smooth affine scheme $\text{Rep}_V(A)$, and the morphism of (ii) is its evaluation morphism.*

Sketch of proof. (i) Given any morphism $\mathcal{F}(A) \rightarrow (B \otimes \text{WEnd}(V))$ of wheelgebras, the restriction to $A \rightarrow (B \otimes \text{End}(V))$ factors uniquely through the standard evaluation morphism $A \rightarrow (\mathbf{k}[\text{Rep}_V(A)] \otimes \text{End}(V))$ via an algebra map $B \rightarrow \mathbf{k}[\text{Rep}_V(A)]$. Since $B \otimes \text{End}(V)$ generates $B \otimes \text{WEnd}(V)$ by wheelgebra operations, the morphism $B \rightarrow \mathbf{k}[\text{Rep}_V(A)]$ must in fact factor the original wheelgebra morphism through the wheelgebra evaluation morphism $\mathcal{F}(A) \rightarrow (\mathbf{k}[\text{Rep}_V(A)] \otimes \text{WEnd}(V))$.

(ii) We have a tautological map of wheelgebras (where \otimes_{wh} is the tensor product in the category of wheelspaces, as in (3.1.8))

$$\mathcal{D}(\mathcal{F}(A)) \otimes_{\text{wh}} \mathcal{F}(A) \rightarrow \mathcal{F}(A), \quad (6.1.9)$$

and can compose with the evaluation morphism to obtain a wheelgebra map $\mathcal{D}(\mathcal{F}(A)) \otimes_{\text{wh}} \mathcal{F}(A) \rightarrow \text{WRep}_V(A)$. We claim that this descends to a morphism $\mathcal{D}(\mathcal{F}(A)) \otimes_{\text{wh}} \text{WRep}_V(A)^{\text{GL}(V)} \rightarrow \text{WRep}_V(A)$. To see this, we note that $\mathbf{k}[\text{Rep}_V(A)]$ can be presented as the free commutative algebra generated by the linear space $A \otimes \text{End}(V)$ subject to the condition that the diagram commutes:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\text{ev}' \otimes \text{ev}'} & (\mathbf{k}[\text{Rep}_V(A)] \otimes \text{End}(V))^{\otimes 2} \\ \downarrow \text{mult} & & \downarrow \text{mult} \\ A & \xrightarrow{\text{ev}'} & \mathbf{k}[\text{Rep}_V(A)] \otimes \text{End}(V). \end{array} \quad (6.1.10)$$

Here, the morphism ev' is obtained from the original morphism $\text{ev} : A \otimes \text{End}(V) \rightarrow \mathbf{k}[\text{Rep}_V(A)]$ using the trace pairing: $\text{ev}'(a) = a \otimes c$, where $c \in \text{End}(V) \otimes \text{End}(V)$ is the canonical element inverse to the trace pairing. In other words, in a basis of V , we can write $\mathbf{k}[\text{Rep}_V(A)]$ as generated by the matrix coefficients a_{ij} of elements $a \in A$, subject to the linearity relation $\lambda a_{ij} + \mu b_{ij} = (\lambda a + \mu b)_{ij}$ and the matrix multiplication relation $\sum_k a_{ik} a_{kj} = a_{ij}$; the map ev' then sends $a \in A$ to the canonical matrix whose i, j -th coefficient is a_{ij} .

As a result, the fact that our original map (6.1.9) is a morphism of wheelgebras says that the morphism descends as desired. Moreover, we can extend the map to all of $\mathcal{D}(\mathcal{F}(A)) \otimes_{\text{wh}} \text{WRep}_V(A)$, using that this is all obtainable from the $\text{GL}(V)$ -invariants by a linear operation of the form

$$\begin{aligned} \text{tr}_E : (\mathbf{k}[\text{Rep}_V(A)] \otimes \text{End}(V)^{\otimes m})^{\text{GL}(V)} &\rightarrow \mathbf{k}[\text{Rep}_V(A)] \otimes \text{End}(V)^{\otimes(m-1)}, \\ T_1 \otimes \cdots \otimes T_m &\mapsto \text{tr}(ET_1)T_2 \otimes \cdots \otimes T_m, \quad E \in \text{End}(V). \end{aligned} \quad (6.1.11)$$

We have to show this is well defined. First, if E is a multiple of the identity, this is a wheelgebra contraction, so the action commutes with our morphism.

Next, if E is traceless, then the target can only be $\mathrm{GL}(V)$ -invariant if it is zero. So, it remains to show that if E is traceless, then the kernel of tr_E is preserved by any wheeled differential operator on $\mathcal{F}(A)$. By the fundamental theorem of invariant theory, this kernel is spanned by elements of the form $\mathrm{ev}(f \otimes 1)$, where $f \in \mathcal{F}_{m-1}(A)$ and $1 \in A$ is the identity. It remains to show that any wheeled differential operator sends an element of the form $f \otimes 1$ to another element of the same form. By induction on order, we can reduce to the case of the element $1 \in A$ itself (setting f to be the identity of $\mathcal{F}(A)$, of degree zero). Suppose such a differential operator ϕ sends 1 to $g \in \mathcal{F}(A)$. Then again by induction on order, $\phi(1 \otimes 1) = 1 \otimes g + g \otimes 1 + h \otimes 1 \otimes 1$ for some $h \in \mathcal{F}(A)$. By the wheelgebra contraction, this implies that $\phi(1) = 2g + h \otimes 1$, and hence that $g = -h \otimes 1$, as desired.

It remains to show that the resulting operations on $\mathrm{WRep}_V(A)$ are indeed wheeled differential operators. This follows by definition.

(iii) By Theorem 3.6.7.(ii) and induction, one can reduce to the statement that $\mathrm{RA}_V(\mathcal{F}(T_A \mathbb{D}\mathrm{er}(A)))$ coincides with $\mathbf{k}[T^* \mathrm{Rep}_V(A)]$. Thanks to part (i), this is the same as showing that $\mathrm{Rep}_V(T_A \mathbb{D}\mathrm{er}(A)) = T^* \mathrm{Rep}_V(A)$. This last equality amounts to showing that $\mathbb{D}\mathrm{er}(A) \otimes_{A^e} (\mathbf{k}[\mathrm{Rep}_V(A)] \otimes \mathrm{End}(V)) \cong \mathrm{Der}(\mathbf{k}[\mathrm{Rep}_V(A)])$, which follows from the fact that $\mathrm{Der}(A, A \otimes A) \otimes_{A^e} A = \mathrm{Der}(A, A)$, since A is projective as an A -bimodule. \square

As before, the above considerations work equally well in the quiver setting, i.e., over the semisimple ground ring \mathbf{k}^I rather than \mathbf{k} . We then deduce

Corollary 6.1.12. *When A is the path algebra of a quiver, the evaluation map $\mathcal{D}(\mathcal{F}(A)) \rightarrow D(\mathrm{Rep}_V(A)) \otimes \mathrm{WEnd}(V)$ restricts, in degree zero, to the map $\mathcal{D}_0(\mathcal{F}(A)) \rightarrow D(\mathrm{Rep}_V(A))^{\mathrm{GL}(V)}$ of [Sch05], cf. Theorem 5.7.1. Moreover, applying this to the odd cotangent bundle, the wheeled BV structure on $\mathcal{F}(T_A \mathbb{D}\mathrm{er}(A))$ (in this case, viewing $\mathbb{D}\mathrm{er}(A)$ as odd), induces the standard one on $\mathrm{RA}_V(\mathcal{F}(T_A \mathbb{D}\mathrm{er}(A))) = \Lambda T_{\mathrm{Rep}_V(A)}$. Equivalently, the wheeled Calabi-Yau structure on A induces the Calabi-Yau structure on $\mathrm{Rep}_V(A)$.*

Sketch of proof. The first statement follows by comparing explicitly the evaluation morphism above with the construction of [Sch05]: the two closely resemble each other. There is an easy dictionary between terminology here, such as the order of composing partial derivatives, with that of [Sch05], in this case, heights of links over the quiver. The second statement then follows by explicitly comparing the formulas for the BV derivative. In some sense, there is little choice for the BV derivative other than the one we made, in order for it to induce the necklace bracket. More precisely, the choice is only one of which connection to use, and the trivial (tautological) one is the choice made here and in [Sch05]. \square

According to Theorem 5.7.1, in this case we also deduce that the action of $\mathcal{D}_0(\mathcal{F}(A))$ on $\mathcal{F}(A)$ is recovered from the action of $D(\mathrm{Rep}_V(A))^{\mathrm{GL}(V)}$ on $\mathrm{Rep}_V(A)$ in the limit as $\dim V \rightarrow \infty$. In the next section, we will generalize this arbitrary commutative wheelgebras.

Sketch of proof of Theorem 6.1.7. To construct $\mathrm{RA}_V(\mathcal{W})$, first form the free associative algebra generated by the vector spaces $\mathcal{W}_m \otimes \mathrm{End}(V)^{\otimes m}$ for all $m \geq 0$. Along with this, as in (6.1.10), form the evaluation morphisms $\mathrm{ev}' : \mathcal{W}_m \rightarrow (\mathcal{W}_m \otimes \mathrm{End}(V)^{\otimes m}) \otimes \mathrm{End}(V)^{\otimes m}$ using the canonical element in $\mathrm{End}(V)^{\otimes m} \otimes \mathrm{End}(V)^{\otimes m}$ inverse to the trace pairing. Then, as we did after (6.1.10), we quotient by the condition that ev' morphism is a map of wheelgebras. For the next statement, the action of $\mathrm{GL}(V)$ is the one induced by the tautological diagonal action on the second factor of the generating spaces $\mathcal{W}_m \otimes \mathrm{End}(V)^{\otimes m}$. To see that the morphism is surjective onto $\mathrm{GL}(V)$ -invariants is an application of the fundamental theorem of invariant theory, similarly to the case $\mathcal{W} = \mathcal{F}(A)$. For the final statement, in the case that \mathcal{W} is (almost) commutative, we deduce that $\mathrm{RA}_V(\mathcal{W}) \otimes \mathrm{WEnd}(V)$ is as well, and hence so is $\mathrm{RA}_V(\mathcal{W})$. For BV or Poisson structures, we have to show that the BV differential or Poisson bracket descends (and extends) to $\mathrm{RA}_V(\mathcal{W})$. This follows in the same manner as in the proof of Theorem 6.1.8.(ii), or alternatively, one can apply the result using that the BV differential is a differential operator of order two, and a Poisson bracket is a differential operator of two inputs, of order one in each input. \square

Remark 6.1.13. The above considerations can also be applied to wheeled PROPs (a generalization of commutative wheelgebras), or even wheeled PROs (the noncommutative analogue of wheeled PROPs). For example, let \mathcal{W} be the wheeled associative PROP, whose representations, over finite-dimensional vector spaces, are the same as finite-dimensional algebras. Then, $\mathrm{RA}_V(\mathcal{W})$ is the structure-constant algebra: in coordinates $V \cong \mathbf{k}^n$, this is the free commutative algebra generated by elements c_{ij}^k for $i, j, k \in \{1, 2, \dots, n\}$, modulo the associativity relation $\sum_{k=1}^n c_{ij}^k c_{kl}^m = \sum_{k=1}^n c_{ik}^m c_{jl}^k$. We interpret the elements c_{ij}^k as structure constants for an associative algebra on \mathbf{k}^n . Then, an associative algebra structure on V over a base ring B is the same as an algebra morphism $\mathrm{RA}_V(\mathcal{W}) \rightarrow B$, which is in turn the same as a representation of the original wheeled PROP in $B \otimes \mathrm{WEnd}(V)$. The same applies to the wheeled commutative, Lie, etc., PROPs.

6.2. The limit $\dim V \rightarrow \infty$. By a *finitely-generated wheelgebra* we mean a wheelgebra which is generated, using wheelgebra operations, by a finite-dimensional \mathbb{S} -module, or equivalently, by a finite-dimensional wheelspace.

Proposition 6.2.1. *Let \mathcal{W} be a finitely-generated wheelgebra. Then, as $d \rightarrow \infty$, the maps $\mathrm{ev}_d : \mathcal{W} \rightarrow \mathrm{WRep}_{\mathbf{k}^d}(\mathcal{W})$ are asymptotically injective: for every finite-dimensional vector subspace U of \mathcal{W} , there exists $N \gg 0$ such that ev_d is injective restricted to U . Hence, $\mathcal{W} \cong \lim_{d \rightarrow \infty} \mathrm{WRep}_{\mathbf{k}^d}(\mathcal{W})^{\mathrm{GL}(\mathbf{k}^d)}$.*

Proof. This follows, as in [Gin01], from the second fundamental theorem of invariant theory, in the same way that the fact that ev is surjective onto the $\mathrm{GL}(V)$ -invariant part is a consequence of the first fundamental theorem. In more detail, recall from the proof of Theorem 6.1.7 that $\mathrm{RA}_V(\mathcal{W})$ is generated by elements of the form $\mathcal{W}_p \otimes \mathrm{End}(V)^{\otimes p}$. Next, restrict ev_d to $M \otimes \mathrm{End}(V)^{\otimes m}$,

where $M \subseteq \mathrm{RA}_V(\mathcal{W})$ is a subspace generated by elements of $\mathcal{W}_p \otimes \mathrm{End}(V)^{\otimes p}$. It follows from the second fundamental theorem of invariant theory that, when $d \geq m + p$, ev_d is injective on $M \otimes \mathrm{End}(V)^{\otimes m}$. On the other hand, \mathcal{W} is linearly spanned by such subspaces, which proves the result. \square

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